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Summary
- Invariant bilinear form
- Cartan's criterion.
  $\mathfrak{g}$ semisimple $\Leftrightarrow$ Killing form is nondegen.

Suppelementary for last time
$\mathrm{ad}\,\mathfrak{g} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ $\left(= \{ \underset{Y \mapsto [X,Y]}{\mathrm{ad}_X} \in \mathrm{End}(\mathfrak{g}) : X \in \mathfrak{g} \}\right)$
is nilpot $\to$ $\mathfrak{g}$ nilpot.

$\mathcal{D}_k(\mathrm{ad}\,\mathfrak{g}) = 0 \Leftrightarrow \mathcal{D}_k(\mathfrak{g}) \subset \mathfrak{z}(\mathfrak{g})$
(img of $\mathcal{D}_k(\mathfrak{g})$)
$\Rightarrow \mathcal{D}_{k+1}(\mathfrak{g}) = [\mathfrak{g}, \mathcal{D}_k(\mathfrak{g})] = 0$

Invariant bilinear forms.

$\mathfrak{g} \overset{\pi}{\curvearrowright} V$ rep.

Def. Invariant bilin. form on V is
$V \times V \to \mathbb{C}$ (or $\mathbb{R}$) (v, w) bilin.
s.t. $(Xv, w) + (v, Xw) = 0$

Why: $(e^{tX} v, e^{tX} w) = (v, w)$
diff. by t $\rightsquigarrow$ get this cond.

Ex. $\mathfrak{g} = \mathfrak{so}_n = \{ X \in M_n(\mathbb{R}),\ X^t = -X \}$
$V = \mathbb{R}^n$ $\rightsquigarrow$ usual inn. prod. on V is
$\mathfrak{so}_n$-inv.
Same w/ complex coeff

Def Adjoint representation $\mathfrak{g} \curvearrowright \mathfrak{g}$
$\mathrm{ad}_X Y = [X, Y]$ satisfies $[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]}$
Jacobi identity.
$\rightsquigarrow$ Lie alg hom $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

$G \overset{\mathrm{Ad}}{\curvearrowright} G$ $\mathrm{Ad}_g(h) = ghg^{-1}$ fixes e, $\rightsquigarrow G \overset{\mathrm{Ad}}{\curvearrowright} T_eG$

Invar. inn. prod. for ad

Ex. $\mathfrak{g} \subset \mathfrak{gl}_n \rightsquigarrow (X, Y) = \mathrm{Tr}(XY)$. trace form

$\because$ From $e^{t\,\mathrm{ad}_X}(Y) = e^{tX}\, Y\, e^{-tX}$

$$(e^{t\,\mathrm{ad}_X}(Y), e^{t\,\mathrm{ad}_X}(Z)) = \mathrm{Tr}(e^{tX} Y Z e^{-tX})$$
$$= \mathrm{Tr}(YZ) = (Y, Z)$$

$\xrightarrow{\text{differentiate}}$ $(\mathrm{ad}_X Y, Z) + (Y, \mathrm{ad}_X Z) = 0.$

Or directly check

$(\mathrm{ad}_X Y, Z) + (Y, \mathrm{ad}_X Z)$

$= \mathrm{Tr}(XYZ - YXZ + YXZ - YZX)$

Ex. <u>Killing form</u>

$B(X, Y) = \mathrm{Tr}(\mathrm{ad}_X\, \mathrm{ad}_Y)$ (trace form on $\mathrm{End}(\mathfrak{g})$.)

Comp. for $\mathfrak{sl}_2$

Ex. $\mathfrak{g} = \mathfrak{sl}_n \rightsquigarrow B(X, Y) = 2n\,\mathrm{Tr}(XY)$

Fact. $\mathfrak{g} \subset \mathfrak{gl}_n$, $\mathrm{Tr}(XY) = 0 \quad \forall X, Y \in \mathfrak{g}$

$\Rightarrow \mathfrak{g}$ is solvable

(Cartan's criterion for solvability)

Thm (Cartan's criterion for semisimplicity)

$\mathfrak{g}$ semisimple $\Leftrightarrow$ Killing form B is nondegenerate.

(Cor. $\mathfrak{sl}_n$ is semisimple)

Proof.

Step 1. $\mathfrak{s} = \{X \in \mathfrak{g} : \forall Y \ B(X, Y) = 0\}$ is an ideal of $\mathfrak{g}$

$\because$ Suppose $X \in \mathfrak{s}$, $Y, Z \in \mathfrak{g}$

Want $[Z, X] \in \mathfrak{s}$ i.e. $B([Z, X], Y) = 0$

Invariance: $B([Z, X], Y) + B(X, [Z, Y]) = 0$

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Step 2 "$\mathfrak{s}$" is solvable

$\because$ $\mathrm{ad}_{\mathfrak{s}} = \{ \mathrm{ad}_X : X \in \mathfrak{s} \} \subset \mathrm{End}(\mathfrak{g})$

$B(\mathrm{ad}_{\mathfrak{s}}, \mathrm{ad}_{\mathfrak{s}}) = 0 \underset{\substack{\text{Criterion} \\ \text{for solvability}}}{\Rightarrow} \mathrm{ad}_{\mathfrak{s}}$ solvable.

ext. $\underset{\text{comm}}{\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{s}} \to \mathfrak{s} \to \underset{\text{solv.}}{\mathrm{ad}_{\mathfrak{s}}} \Rightarrow \mathfrak{s}$ solvable

Step 3 "$\Rightarrow$" of Th'm.

$\mathfrak{g}$ semisimple $\Rightarrow$ no solvable ideal

$\Rightarrow \forall X \neq 0 \ \exists Y : B(X, Y) \neq 0$

Step 4. $\mathfrak{h} \triangleleft \mathfrak{g} \Rightarrow \mathcal{D}(\mathfrak{h}) \triangleleft \mathfrak{g}$.

$\because$ Jacobi id.

Step 5 $\mathrm{Rad}(\mathfrak{g}) \neq 0 \Rightarrow \exists$ nonzero comm. ideal.

$\because$ $\mathrm{Rad}(\mathfrak{g})$ is solvable by def.

$\mathcal{D}^k(\mathrm{Rad}(\mathfrak{g})) \triangleleft \mathfrak{g}$ by step 4

$\mathcal{D}^{k+1}(\mathrm{Rad}(\mathfrak{g})) = \mathcal{D}(\mathcal{D}^k(\mathrm{Rad}(\mathfrak{g}))) = [\mathcal{D}^k(\mathrm{Rad}(\mathfrak{g})), \mathcal{D}^k(\mathrm{Rad}(\mathfrak{g}))]$

$= 0$

$\Rightarrow \mathcal{D}^k(\mathrm{Rad}(\mathfrak{g}))$ comm.

Step 6 "$\Leftarrow$" of Th'm.

$\because$ Want : B nondeg $\Rightarrow$ no comm. ideal

(enough by Step 5)

Suppose $\mathfrak{a} \triangleleft \mathfrak{g}$ comm, $X \in \mathfrak{a}$, $Y \in \mathfrak{g}$

$\mathrm{ad}_X \mathrm{ad}_Y : \mathfrak{g} \to \mathfrak{a}$ by $\mathrm{ad}_X(\cdot)$

$\mathfrak{a} \to 0$ by $\mathrm{ad}_Y(\mathfrak{a}) \subset \mathfrak{a}$, $\mathrm{ad}_X(\mathfrak{a}) = 0$.

So mat form is $\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \begin{array}{l} \} \mathfrak{a} \\ \} \mathfrak{g} \ominus \mathfrak{a} \end{array}$

$\rightsquigarrow \underbrace{Tr_{End(\mathfrak{g})}(ad_X\, ad_Y)}_{B(X,Y)} = 0 \underset{\text{nondegeneracy}}{\Rightarrow} X = 0$ ☐

Criterion for solvability

$\mathfrak{g} \subset \mathfrak{gl}_n$, $Tr(\mathfrak{g}\mathfrak{g}) = 0 \Rightarrow \mathfrak{g}$ solv.

Idea $\mathcal{D}(\mathfrak{g})$ nilpot $\Rightarrow \mathfrak{g}$ solv.

$\mathcal{D}(\mathfrak{g}) \to \mathfrak{g} \to \underset{\text{comm}}{\mathfrak{g}/\mathcal{D}(\mathfrak{g})}$

Step 1. Engel's thm $\Rightarrow$ enough to check each $X \in \mathcal{D}(\mathfrak{g})$ is nilpot. mat.

eigenvals are all 0

Step 2 Take $X \in \mathfrak{gl}_n$, write

$X = \underset{\text{diag'bl}}{X_s} + \underset{\text{nilpot}}{X_n}$ $\quad X_s \sim \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ (eigenvals)

$X_n \sim \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$

$\bar{X}_s$ = "conjugate" of X_s

$\Rightarrow Tr(\bar{X}_s X) = \sum_{i=1}^{n} |\lambda_i|^2 \leftarrow$ want to be 0 for $X \in \mathcal{D}(\mathfrak{g})$

Step 3 $ad_{\bar{X}_s}$ is polynom. in ad_X

$\because$ $ad_{X_s} = (ad_X)_s$ $\quad ad_{X_n}$ is nilpot, comm. w/ ad_{X_s}.

$(ad_X)_s$ is polynom. in ad_X

$ad_{\bar{X}_s} = \overline{(ad_X)_s}$ polynom. in $(ad_X)_s$

Step 4. Claim of Step 1.

$\because$ $X = \sum_{i=1}^{k} [Y_i, Z_i]$

$Tr(\bar{X}_s X) \underset{\text{invar.}}{=} \sum Tr(\underbrace{[\bar{X}_s, Y_i]}_{ad_{\bar{X}_s}(Y_i) \underset{\text{Step 3}}{\in} \mathfrak{g}} Z_i) \overset{\text{assumption}}{=} 0$