Summary

- Complete reducibility
- Semisimple - nilpot. decomp. in semisimple Lie algs.

Complete reducibility

Want : (π, V) rep of $\mathfrak{g}$

$\rightsquigarrow$ $(\pi, V) \simeq (\pi_1, V_1) \oplus \cdots \oplus (\pi_n, V_n)$

(π_i, V_i) irreducible reps.

$\rightsquigarrow$ will be enough to understand irreps. of $\mathfrak{g}$

What can go wrong : $\mathfrak{g} = \mathbb{R}$

rep of $\mathfrak{g}$: same as specifying

V, $T = \pi_1 \in \mathrm{End}(V)$ so $\pi_r = rT$ for $r \in \mathbb{R} = \mathfrak{g}$.

$\to$ Jordan normal form of T

$$T \leftrightarrow \begin{bmatrix} \lambda_1 & 1 & & & & \\ & \ddots & 1 & & & \\ & & \lambda_1 & 0 & & \\ & & & \lambda_2 & 1 & \\ & & & & & \ddots \end{bmatrix} \quad \text{for some basis}$$

$\begin{bmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ would be invariant subsp.

but w/o complement in gen.

e.g. inv. subsp. for $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$: $\{0\}$, $\underbrace{\left\{ \begin{bmatrix} z \\ 0 \end{bmatrix} : z \in \mathbb{C} \right\}}_{\text{no compl.}}$, $\mathbb{C}^2$

Th'm $\mathfrak{g}$ semisimple Lie alg., (π, V) rep. of $\mathfrak{g}$

$W \subset V$: $\mathfrak{g}$-inv. subsp. $\rightsquigarrow$ $\exists$ $\mathfrak{g}$ inv. subsp. $W' \subset V$ s.t. $V = W \oplus W'$

Proof. Step 0 WMA $\mathfrak{g} \subset \mathfrak{gl}(V)$

Step 1. $B_V(X,Y) = \mathrm{Tr}_V(XY)$ $\quad X,Y \in \mathfrak{gl}(V)$

is nondegenerate on $\mathfrak{g}$

$\because$ $\mathfrak{s} = \{X \in \mathfrak{g} : \forall Y \in \mathfrak{g} \quad B_V(X,Y) = 0\}$

invariance of $B_V \Rightarrow \mathfrak{s}$ is an ideal $\triangleleft\, \mathfrak{g}$

Cartan's criterion $\Rightarrow \mathfrak{s}$ is solvable.

$\underset{\mathfrak{g}\text{ is s.s}}{\Longrightarrow} \mathfrak{s} = 0$

Step 2 $\quad (X_i)_{i=1}^n$ basis of $\mathfrak{g}$,

$(X^i)_{i=1}^n$ <u>dual basis</u> w.r.t. B_V $\quad (B_V(X_i, X^j) = \delta_{ij})$

consider Casimir operator $C_V = \sum_i X_i X^i \in \mathrm{End}(V)$

$\Rightarrow C_V$ is an intertwiner.

$\because$ $[Y, C_V] = \sum_i [Y, X_i] X^i + X_i [Y, X^i]$

$= \sum_{i,j} B_V([Y,X_i], X^j)\, X_j X^i + B_V(X_j, [Y,X^i])\, X_i X^j$

switch role of i & j, use invariance $\rightsquigarrow 0$

Step 3 $\quad$ Claim for $\dim W = \dim V - 1$, W irred.

$\because$ • $C_V|_W$ is scalar by Schur's lem.
& irred. of W.

• $\mathfrak{g} \curvearrowright V/W \in$ 1-dim is triv
$(\mathfrak{g} = \mathcal{D}(\mathfrak{g})$ should act by $0)$.

• $\mathrm{Tr}_V(C_V) = \sum_i \mathrm{Tr}(X_i X^i) = \sum_i B_V(X_i, X^i) = n$
$(= \dim \mathfrak{g})$

$\rightsquigarrow C_V|_W$ is nonzero $\dfrac{\dim \mathfrak{g}}{\dim W} \mathrm{Id}_W$.

$C_V(V) \subset W$ $\quad (\overline{C_V} = 0 \in \mathrm{End}(V/W).)$

so $\dfrac{\dim W}{\dim \mathfrak{g}} C_V$ is a projector to W

$V = W \oplus \mathrm{Ker}\, C_V$.

We do the rest by ind. on $\dim V$

Step 4. Claim for $\dim W = \dim V - 1$.

$\because$ induction on $\dim W$.

If $\exists\ 0 \neq Z \subset W$ invar.

$$V/Z \simeq \underbrace{W/Z}_{\text{dim smaller than } W} \oplus Y \qquad \dim Y = 1$$

Inv. img $\tilde{Y} \subset V$ of Y contains Z.

By ind. hypo on $\dim V$ ($> \dim \tilde{Y}$)

$\tilde{Y} = Z \oplus Y' \Rightarrow V \simeq W \oplus Y'$

Step 5 $\dim W$ general, W irred.

$\because$ $\mathrm{End}_{\mathfrak{g}}(W)$ is 1-dim triv. rep of $\mathfrak{g}$

$\mathrm{Hom}(V, W) \xrightarrow{\text{res}} \mathrm{End}(W)$ surj, $\mathfrak{g}$-equivar

$\cup \qquad\qquad \cup$

take inv. img $U \to \mathrm{End}_{\mathfrak{g}}(W)$

$\hookrightarrow$ $\mathrm{Ker}(\mathrm{res}|_U)$ has codim 1 (and $\mathfrak{g}$-inv)

$\underset{\text{Step 4}}{\Longrightarrow}$ $U = \mathrm{Ker}(\mathrm{res}|_U) \oplus \underset{\text{triv. rep.} \subset \mathrm{Hom}_{\mathfrak{g}}(V, W)}{U_0}$

So $\exists$ $\mathfrak{g}$-intertw. $T : V \to W$ s.t. $T|_W = \mathrm{Id}_W$.

$\Rightarrow V = \mathrm{Ker}\, T \oplus W$.

Step 6 General : same as Step 4. $\square$

- Semisimple - nilpotent decomposition

Recall $T \in \mathrm{End}(V)$ Jordan normal form

$$\begin{bmatrix} \lambda_1 & 1 & & & & \\ & \lambda_1 & \ddots & & & \\ & & \ddots & 1 & & \\ & & & \lambda_1 & 0 & \\ & & & & \lambda_2 & 1 \\ & & & & & \ddots \end{bmatrix} \rightsquigarrow T_s \leftrightarrow \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_1 & \\ & & & \lambda_2 \\ & & & & \ddots \end{bmatrix}}_{\text{"semisimple part"}}, \quad T_n \leftrightarrow \underbrace{\begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & 0 \\ & & & & 0 & 1 \\ & & & & & \ddots \end{bmatrix}}_{\text{nilpotent part}}$$

$T = T_s + T_n$ diagonalizable $\longleftrightarrow$ commute

Want : this decomp. for π_X $X \in \mathfrak{g}$.

Prob : generally $\nexists\, Y, Z \in \mathfrak{g}$ s.t.

$(\pi_X)_s = \pi_Y$, $(\pi_X)_n = \pi_Z$

e.g. $\mathfrak{g} = \mathbb{R}$, $\pi_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Prop. $\mathfrak{g} \subset \mathfrak{gl}(V)$ semisimple , $X \in \mathfrak{g}$

$\Rightarrow X_s, X_n \in \mathfrak{g}$

Cor semisimple $\mathfrak{g}$, $X \in \mathfrak{g}$

$\Rightarrow \exists\, Y, Z \in \mathfrak{g}$ $\forall \pi$: $\pi_Y = (\pi_X)_s$, $\pi_Z = (\pi_X)_n$

Proof of Prop.

Step 1 $\mathfrak{g} = \tilde{\mathfrak{g}} \cap \left(\bigcap_{W : \mathfrak{g}\text{-inv.}} \mathfrak{s}_W \right)$ with

$\tilde{\mathfrak{g}} = \{ Y \in \mathfrak{gl}(V) : [Y, \mathfrak{g}] \subset \mathfrak{g} \}$ normalizer

$\mathfrak{s}_W = \{ Y \in \mathfrak{gl}(V) : YW \subset W , \operatorname{Tr}(Y|_W) = 0 \}$

$\because$ $\subset$ follows from $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, etc.

$\supset$: Put $\mathfrak{g}' = \tilde{\mathfrak{g}} \cap \left(\bigcap_W \mathfrak{s}_W \right)$

Look at $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}'$ $\xrightarrow[\text{comp. red}]{}$ $\mathfrak{g}' = \mathfrak{g} \oplus \underset{\text{ad-inv}}{U}$

Want $U = 0$; take $Y \in U$

Enough to show : $W \subset V$ irred subrep. of $\mathfrak{g}$ $\Rightarrow Y|_W = 0$

$[Y, \mathfrak{g}] \in \mathfrak{g} \cap U = 0 \Rightarrow Y$ is intertw.

$\Rightarrow Y|_W$ is scalar, $\underset{\operatorname{Tr}(Y|_W) = 0}{= 0}$

Goal : $X_s \in \tilde{\mathfrak{g}}$, $X_s \in \mathfrak{s}_W$ separately.

Step 2 $X_s, X_n \in \mathfrak{s}_W$ for $\mathfrak{g}$-inv $W \subset V$

$\because$ $X = \sum [Y_i, Z_i]$ by $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$

$\rightsquigarrow \operatorname{Tr}(X|_W) = \sum \operatorname{Tr}(Y_i|_W, Z_i|_W) = 0$

$X_n|_W$ nilp. $\Rightarrow$ $\mathrm{Tr}(X_n|_W) = 0$

$\mathrm{Tr}(X_s|_W) = \mathrm{Tr}(X|_W - X_n|_W) = 0$

Step 3 $X_s, X_n \in \tilde{\mathfrak{g}}$

$\because$ ad_{X_s}, ad_{X_n} are semisimple / nilpot parts of ad_X $\Rightarrow$ polynoms in ad_X

$\mathrm{ad}_X(\mathfrak{g}) \subset \mathfrak{g}$ $\Rightarrow$ $\mathrm{ad}_{X_s}(\mathfrak{g}) \subset \mathfrak{g}$, etc.