Summary.

(Semi)simple Lie algebras / $\mathbb{C}$.

- Cartan subalgs
- root system.

§ Classification of simple Lie alg's / $\mathbb{C}$

$\mathfrak{g}$ semisimple $\Rightarrow$ $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$

$\mathfrak{g}_i$: simple (no nonzero ideal)

$\rightsquigarrow$ We want to understand <u>simple</u> Lie alg's.

Ex. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ is simple.

How to see : put $\mathfrak{h} = \{X \in \mathfrak{g} : \text{diag.}\}$

$X, Y \in \mathfrak{h} \Rightarrow \mathrm{ad}_X, \mathrm{ad}_Y$ comm. diagonalizable.

$\rightsquigarrow$ we should find joint eigenvecs.

$$E_{ij} = \begin{array}{c} \begin{bmatrix} 0 & \cdots & & \\ \vdots & & 1 & \cdots \\ & & & \end{bmatrix} \begin{array}{c} \\ i \\ \\ \end{array} \\ j \end{array} \quad \text{mat. unit.}$$

$\mathfrak{h}$: span of $H_i = E_{ii} - E_{(i+1)(i+1)}$ $(i = 1, \cdots, n-1)$

$\mathrm{ad}_{H_i}(E_{jk}) = [H_i, E_{jk}] = (\delta_{i,j} - \delta_{i+1,j} - \delta_{k,i} + \delta_{k,(i+1)}) E_{jk}$

i.e. $(E_{jk})_{j,k}$ are eigenvecs for ad_{H_i}

$(E_{j,k})_{j \neq k}, (H_j)_j$ basis of $\mathfrak{g}$ by joint eigenvecs.

$0 \neq \mathfrak{g}' \triangleleft \mathfrak{g}$ $\underset{\text{by } \mathfrak{h}\text{-invar.}}{\rightsquigarrow}$ $\exists\, j,k$ $E_{j,k} \in \mathfrak{g}'$ or $\exists\, 0 \neq X \in \mathfrak{h} \cap \mathfrak{g}'$

$[E_{i,j}, E_{j,k}] = E_{i,k}$ etc. $\Rightarrow \mathfrak{g}' = \mathfrak{g}$. $\square$

Def. <u>Cartan subalg</u> of $\mathfrak{g}$ is <u>maximally commutative</u> subalg $\mathfrak{h}$.

s.t. ad_X is diagonalizable for $X \in \mathfrak{h}$.

Rem. $X \in \mathfrak{h} \Rightarrow \pi_X$ is diagonalizable

for any rep. (π, V)

$\because \pi(X)_s = \pi(Y)$ by last time, $\mathrm{ad}_{\pi(X)}|_{\pi(\mathfrak{g})} = \mathrm{ad}_{\pi(Y)}|_{\pi(\mathfrak{g})}$)

$\rightsquigarrow$ V has a basis by joint eigenvecs.

$v_1, \dots, v_m$, $\pi_X v_i = \lambda_i(X) v_i$ $X \in \mathfrak{h}$

$\lambda_i : \mathfrak{h} \to \mathbb{C}$ is linear.

$\lambda_1, \dots, \lambda_m$: weights of (π, V).

Def. $\mathfrak{g}$ (semi) simple Lie a's, $\mathfrak{h} < \mathfrak{g}$ Cartan.

Roots of $\mathfrak{g}$: nonzero weights for $\mathrm{ad}\ \mathfrak{g} \curvearrowright \mathfrak{g}$

(w.r.t. $\mathfrak{h}$)

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha:\text{root}} \mathfrak{g}_\alpha\Big) \quad \text{root decomp.}$$

so $\mathfrak{g}_\alpha = \{ Y \in \mathfrak{g} : [X, Y] = \alpha(X) Y \ \ \forall X \in \mathfrak{h} \}$

Ex. $\mathfrak{sl}_2(\mathbb{C}) = \underset{\text{"}\mathfrak{h}\text{"}}{\mathbb{C}H} \oplus \underset{\text{"}\mathfrak{g}_\alpha\text{"}}{\mathbb{C}E} \oplus \underset{\text{"}\mathfrak{g}_{-\alpha}\text{"}}{\mathbb{C}F}$ $\qquad \alpha(H) = 2$

$[H, E] = 2E$, $[H, F] = -2F$.

Fin. dim irreps $V_n = \langle x^n, x^{n-1}y, \dots, y^n \rangle$

(10.01) weights of V_n: $\lambda_k(H) = k$

$k = n, n-2, \dots, -n$.

Moral :

- weights of reps form a lattice in $\mathfrak{h}^*$
  (like $\mathbb{Z}^n < \mathbb{C}^n$)
- Killing form $B_{\mathfrak{g}}(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y)$
  induces an inner prod.
- roots $\subset$ (weight lattice, inn. prod)
  is a complete invariant of $\mathfrak{g}$

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Overview of simple Lie alg $\mathfrak{g}$ $\rightsquigarrow$ root system

α: root for $(\mathfrak{g}, \mathfrak{h})$ Ex. for $\mathfrak{sl}_n$

Fact 1 $\dim \mathfrak{g}_\alpha = 1$ $(\mathfrak{g}_\alpha = \mathbb{C} E_{j,k})$

Fact 2 $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$

is a subalg of $\mathfrak{g}$, $\mathfrak{s}_\alpha \simeq \mathfrak{sl}_2$.

$(\langle E_{j,k}, E_{k,j}, E_{j,j} - E_{k,k} \rangle \simeq \mathfrak{sl}_2)$

choose $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$, $H_\alpha = [E_\alpha, F_\alpha]$

s.t. $[H_\alpha, E_\alpha] = 2E_\alpha$, $[H_\alpha, F_\alpha] = -2F_\alpha$.

<u>Weight</u> <u>lattice</u> $\Lambda_W = \{ \beta \in \mathfrak{h}^* : \beta(H_\alpha) \in \mathbb{Z}$ for all roots $\alpha \}$

(By Ex. for $\mathfrak{sl}_2$, weights of $(\pi, V) \subset \Lambda_W$).

<u>root</u> <u>lattice</u> Λ_R : span of roots $\subset \Lambda_W$

$E = \mathbb{R} \cdot \Lambda_W \subset \mathfrak{h}^*$, $\mathfrak{h}_0 = \langle H_\alpha : \alpha \text{ roots} \rangle \subset \mathfrak{h}$

Killing form $B_\mathfrak{g}$

- nondeg. on $\mathfrak{g}$ (Cartan's criterion)
- invariant $B_\mathfrak{g}([Z,X],Y) + B_\mathfrak{g}(X,[Z,Y]) = 0$

$\rightsquigarrow$ ($\cdot$ $B_\mathfrak{g}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) \neq 0$ iff $\beta = -\alpha$

$\cdot$ nondeg. on $\mathfrak{h}$ = "$\mathfrak{g}_{\alpha=0}$")

Fact 3 $T_\alpha = \dfrac{2}{B_\mathfrak{g}(H_\alpha, H_\alpha)} H_\alpha$ satisfies

$B_\mathfrak{g}(T_\alpha, X) = \alpha(X)$ for $X \in \mathfrak{h}$.

i.e. $\alpha \in \mathfrak{h}^* \xleftrightarrow[\text{iso. by } B_\mathfrak{g}]{} T_\alpha \in \mathfrak{h}$

so $(\alpha, \beta) := B_\mathfrak{g}(T_\alpha, T_\beta)$

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} = \frac{2B_\mathfrak{g}(T_\alpha, T_\beta)}{B_\mathfrak{g}(T_\beta, T_\beta)} = B_\mathfrak{g}(T_\alpha, H_\beta) = \alpha(H_\beta) \in \mathbb{Z}.$$

Fact 4 $\quad B_{\mathfrak{g}}(X, X) \geq 0$ for $X \in \mathfrak{h}_0$.

$\qquad = 0 \iff X = 0$.

$\because \quad X = \sum_i r_i H_{\alpha_i} \qquad r_i \in \mathbb{R}$

acts by $\sum_i r_i \beta(\alpha_i)$ on $\mathfrak{g}_\beta$

0 on $\mathfrak{h}$.

$\Rightarrow \mathrm{Tr}\left((\mathrm{ad}_X)^2\right) = \sum_\beta \left(\sum_i r_i \beta(\alpha_i)\right)^2$

(α, β) Euclidean inn. prod.

Fact 5 $\quad \alpha$ root, $k = \pm 2, \pm 3, \cdots \Rightarrow k\alpha$ not root.

Ex. Roots of $\mathfrak{sl}_3(\mathbb{C})$

$\mathfrak{h} = \langle H_1 = E_{11} - E_{22},\ H_2 = E_{22} - E_{33} \rangle$

roots α_1 : e-val for e-vec E_{12}

α_2 ~~~ E_{23}.

$[H_1, E_{12}] = 2E_{12}, \quad [H_2, E_{12}] = -E_{12}$

$\Rightarrow \alpha_1(H_1) = 2, \quad \alpha_1(H_2) = -1$

similarly $\alpha_2(H_1) = -1, \quad \alpha_2(H_2) = 2$

other roots : for E_{13} : $\alpha_1 + \alpha_2$

$\because [H_1, E_{13}] = E_{13} = [H_2, H_{13}]$

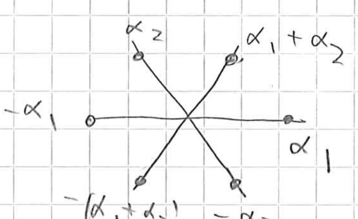
$-\alpha_1, -\alpha_2, -(\alpha_1 + \alpha_2)$

$B_{\mathfrak{g}}(X, Y) = 6\,\mathrm{Tr}(XY)$ for $\mathfrak{sl}_3 = \mathfrak{g}$

$\rightsquigarrow B_{\mathfrak{g}}(H_i, H_i) = 12, \quad B_{\mathfrak{g}}(H_1, H_2) = -6$

$\rightsquigarrow T_{\alpha_i} = \frac{1}{6} H_i \quad i = 1, 2 \qquad (\alpha_1, \alpha_2) = -\frac{1}{2}(\alpha_1, \alpha_1)$

$(\alpha_2, \alpha_2) = (\alpha_1, \alpha_1)$



Fact 6 $\quad R = \{\alpha : \text{roots}\}$ span $E = \mathbb{R}\Lambda_w$.

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$$s_\alpha(v) = v - \frac{2(\alpha, v)}{(\alpha,\alpha)}\alpha \quad \text{for } v \in E$$

reflection along $\alpha^\perp = \{ w \in E : (\alpha, w) = 0 \}$.

(π, V) rep. of $\mathfrak{g}$. β weight.

$E_\alpha V_\beta \subset V_{\beta+\alpha}$, $F_\alpha V_\beta \subset V_{\beta-\alpha}$

$\Rightarrow$ $\mathrm{Supp}(V) = \{\beta : \text{wght of } V\}$

is inv. under s_α.

$\Rightarrow$ R is inv. under s_α.