Summary
- Axiomatization of root system
- Dynkin diagrams.

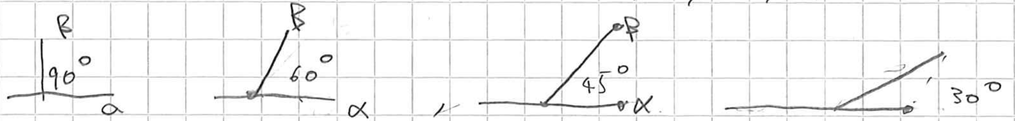
Def. A root system is given by
- Euclidean space E
- fin. subset $R \subset E$ s.t.

1) R spans E
2) $\alpha \in R$, $k \in \mathbb{Z}$: $k\alpha \in R \Leftrightarrow k = \pm 1$
3) $s_\alpha(v) = v - \dfrac{2(\alpha, v)}{(\alpha,\alpha)}\alpha$ $(\alpha \in R)$ maps R to R
4) $n_{\alpha,\beta} = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

Prop $\alpha, \beta \in R \Rightarrow$ angles between α & β is $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$, $\pi -$ (one of these)

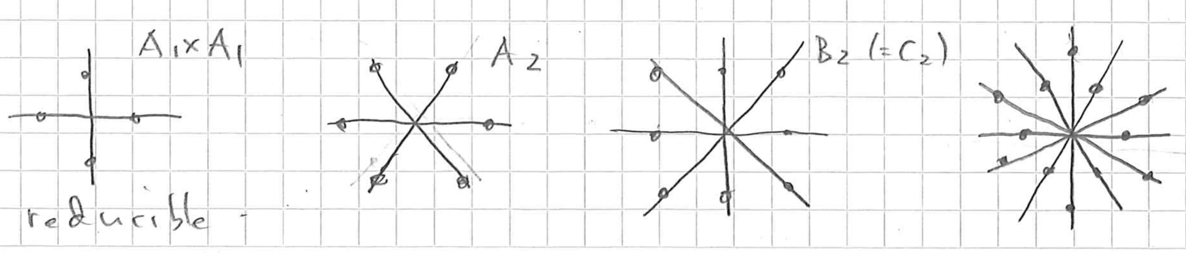
Proof $n_{\alpha,\beta}\, n_{\beta,\alpha} = 4\dfrac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = 4\cos^2\theta$ for the angle θ.

$\Rightarrow \cos\theta = \pm\dfrac{\sqrt{a}}{2}$ for $a \in \mathbb{N}$ only possible for $a = 0, 1, 2, 3$.



$\|\alpha\| = \|\beta\|$ $\|\beta\| = \sqrt{2}^{\pm 1}\|\alpha\|$ $\|\beta\| = \sqrt{3}^{\pm 1}\|\alpha\|$

Ex. If $\dim E = 2$ R is one of :



Rem $(E_1, R_1), (E_2, R_2)$ root sys $\Rightarrow (E_1 \times E_2, R_1 \times R_2)$

Def <u>irred.</u> root sys: $\nexists$ decomp. as above.

So far: $\mathfrak{g}$ simple Lie alg., $\mathfrak{h} \subset \mathfrak{g}$ Cartan

$\longrightarrow$ root system by

- R: roots (nonzero eigenval funcs. of $\mathrm{ad}_{\mathfrak{h}}$)

  so $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$

- $E = \mathbb{R}\Lambda_W = \mathbb{R}\Lambda_R$ real span of R $\subset \mathfrak{h}^*$

  inner prod. $\longleftrightarrow$ Killing form on.

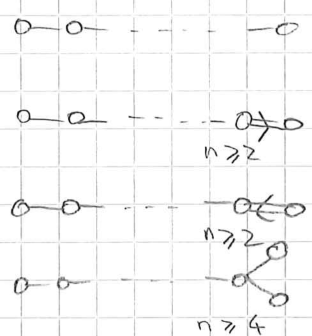
  $E^* = \mathfrak{h}_0 = \langle H_\alpha = [E_\alpha, F_\alpha] : \alpha \in R \rangle_{\mathbb{R}}$

Goal:

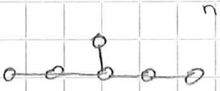
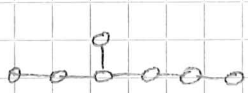
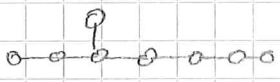
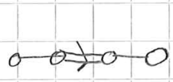
1. Classify irred. root systems by Dynkin <u>diagrams</u> $(E, R) \rightsquigarrow$ diag.
2. check that irred. root sys. come from simple Lie algs

   diag $\rightsquigarrow (\mathfrak{g}, \mathfrak{h})$ ($\rightsquigarrow$ "root sys" will be inverse op.)

<u>Dynkin diagrams</u>

- A_n  $\mathfrak{sl}_{n+1}$
- B_n $n \geq 2$ $\mathfrak{so}_{2n+1}$
- C_n $n \geq 2$ $\mathfrak{sp}_{2n}$
- D_n $n \geq 4$ $\mathfrak{so}_{2n}$

$\left.\right\}$ n vertices

- E_6  • E_7 
- E_8  • F_4  • G_2