2019.11.06. no. 1

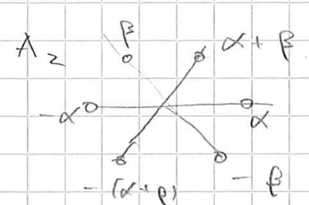
Summary

- irred. root sys $\longleftrightarrow$ Dynkin diagram.
  - positive roots.
  - Weyl group.

(E, R) (irred.) root system.

Want : good subset $\Pi \subset R$

- large enough : can span E.
- small enough : no redundancy.



enough to know α, β.

Ex. $\mathfrak{sl}_{n+1} \rightsquigarrow$ n-dim root sys. (E, R)

$R = \{ \alpha_{i,j}$: e-val func. for (i,j)-th mat. unit $E_{ij} \}$

$\Pi = \{ \alpha_{i,i+1} : i = 1, \ldots, n-1 \}$ "just above diag."

Abstractly : divide R as $R^+ \amalg R^-$
"positive" / "negative" roots

by taking $\ell : E \to \mathbb{R}$ linear, $\ker \ell \cap R = \varnothing$.

$R^+ = \{ \alpha \in R, \ell(\alpha) > 0 \}$, $R^- = \{ \alpha \in R, \ell(\alpha) < 0 \}$

Def. $\alpha \in R^+$ is <u>simple</u> if $\nexists\, \beta, \gamma \in R^+$ $\alpha = \beta + \gamma$.

$\Pi = \{$ simple pos. roots $\}$

Ex. $\mathfrak{sl}_3 \rightsquigarrow$ $R = \{ \underbrace{\alpha_{12}, \alpha_{23}, \alpha_{13}}_{\text{pos.}}, \underbrace{\alpha_{21}, \alpha_{32}, \alpha_{31}}_{\text{neg.}} \}$

$\alpha_{13} = \alpha_{12} + \alpha_{23}$: $[E_{12}, E_{23}] = E_{13}$.

$\Rightarrow [H, [E_{12}, E_{23}]] = [[H, E_{12}], E_{23}] + [E_{12}, [H, E_{23}]]$
Jacobi & antisym.

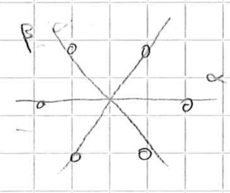
$\Rightarrow \alpha_{12}, \alpha_{23}$ are the simple pos. roots.

How to draw a Dynkin diagram.

- vertices : simple pos. rts. $\Pi = \{\alpha_1, \dots, \alpha_n\}$
- edges between $\alpha, \beta \in \Pi$: k edges

if $\frac{(\alpha, \beta)}{\|\alpha\| \cdot \|\beta\|} = \text{"}\cos\theta\text{"} = \underbrace{\pm}_{\text{will be } -} \frac{\sqrt{k}}{2}$, $(k = 0, 1, 2, 3)$

$k = 2, 3 \Rightarrow$ $\underset{\alpha}{\circ} \Rightarrow \underset{\beta}{\circ}$ etc. if $\|\alpha\| > \|\beta\|$.

Ex.  $\rightsquigarrow$ $\underset{\alpha}{\circ} - \underset{\beta}{\circ}$

Def. Weyl group of (E, R) is

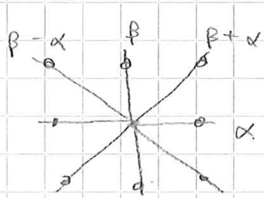
$W = \langle s_\alpha : \alpha \in R \rangle \subset O(E)$ $\qquad s_\alpha(v) = v - \underbrace{\frac{2(\alpha, v)}{(\alpha, \alpha)}}_{n_{\alpha,\beta}} \alpha$

Lem. $\alpha, \beta \in R$ $\alpha \neq \pm\beta$ $p, q = \max$ s.t.

$\beta - p\alpha, \beta - p\alpha + \alpha, \dots, \beta, \beta + \alpha, \dots, \beta + q\alpha$ are

roots ("α-string through β")

Then $p + q \leq 3$, $p - q = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = n_{\alpha, \beta}$.

Ex. B_2 

Proof. $s_\alpha(\beta + k\alpha) = \beta + k\alpha - \frac{2(\alpha, \beta + k\alpha)}{(\alpha, \alpha)} \alpha$

$= \beta - (n_{\alpha,\beta} + k)\alpha$

$\rightsquigarrow s_\alpha$ preserves α-string, $p = n_{\alpha,\beta} + q$

with $\beta' = \beta - p\alpha$ $\quad p + q = n_{\beta', \alpha} \leq 3$.

Prop. 1) $(\alpha, \beta) > 0 \Rightarrow \alpha - \beta$ root

2) $(\alpha, \beta) < 0 \Rightarrow \alpha + \beta$ root

Pf 2) follows from 1 by $\beta' = -\beta \in R$.

1 : $s_{\alpha}\beta = \beta - n_{\alpha,\beta}\alpha \in R \Rightarrow \beta - \alpha$ sits between this & β

Cor $\alpha, \beta \in \Pi$ (simple pos.)

$\Rightarrow (\alpha, \beta) \leq 0$

Step 1. $\alpha - \beta$, $\beta - \alpha$ are <u>not</u> roots

Step 2 Claim. $\because$ Prop. 1.