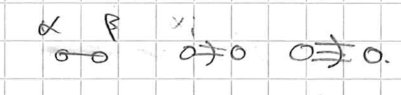
Summary

- classification of Dynkin diagrams
- Lie alg. rel. from Dynkin diagrams

Classification of Dynkin diag.

(irred.) root sys. (E, R) — diagram Γ

$\searrow R = R^+ \sqcup R^- \longrightarrow$ simple pos. roots Π

vertex set Π, edges 

from $n_{\alpha,\beta} = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)}$

Want: Γ is one of X_n $X = A, B, C, D$, $n \geq 1$

(#) E_k $k = 6, 7, 8$, F_4, G_2.

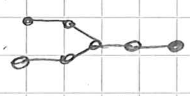
For $\alpha \in \Pi$ put $e_\alpha = \dfrac{1}{\sqrt{(\alpha,\alpha)}}\alpha$ (unit vec.)

so $(e_\alpha, e_\beta) = -\dfrac{\sqrt{d}}{2}$ $d = 0, 1, 2, 3$

d: numbe of edges between α & β.

Thm 1) $\nexists$ loop in Γ

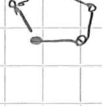
2) each $\alpha \in \Pi$ is connected to ≤ 3 edges

3) $\nexists$  $\subset \Gamma$.

Rem) "the rest" are similar $\rightsquigarrow$ Γ is one of above in (#)

Proof 1) claim $\Leftrightarrow$ $\forall J \subset \Pi$ $\exists$ at most $2(|J|-1)$

pairs $(\alpha, \beta) \in \Pi \times \Pi$ s.t. α & β are connected

($\times$  5 combinations $\times$ 2 orientations

Write $v = \sum_{\alpha \in J} e_\alpha \Rightarrow$

$\rightsquigarrow (v, v) = \sum_{\alpha \in J} \underbrace{(e_\alpha, e_\alpha)}_{1} + \sum_{\substack{\alpha, \beta \in J \\ \alpha \sim \beta}} (e_\alpha, e_\beta)$

$\leq |J| + \quad (\#(\alpha,\beta) \text{ as above}) \times \underbrace{(-\tfrac{1}{2})}_{\text{bounds } (\alpha,\beta)}$

(v, v) is pos. $\quad (\#(\alpha,\beta) \text{ as abv}) < 2|J|$
(even.)

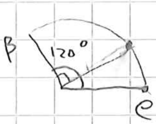
2) Fix $\alpha \in \Pi$. We want

$$\sum_{\beta \in \Pi} \underbrace{\#(\text{edges between } \alpha \,\&\, \beta)}_{4(e_\alpha, e_\beta)^2} < 4 \quad \cdots (*)$$

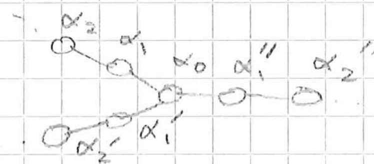
Step 1 $\quad \underset{(\exists \text{ edge})}{\alpha \sim \beta},\ \alpha \sim \gamma \Rightarrow \beta \not\sim \gamma$

Step 2 $(*)$: $(e_\beta)_{\beta \sim \alpha}$ are orth. by Step 1, $\sum_{\beta \sim \alpha} (e_\alpha, e_\beta)^2$: sq. len of proj. of e_α to the span of $(e_\beta)_{\beta \sim \alpha}$

$\rightsquigarrow$ should be less than $\|e_\alpha\|^2 = 1$.

3) 

$\Rightarrow \frac{1}{\sqrt{3}}(2e_\alpha + e_\beta)$ is also unit vec

Suppose we had 

$\subset \Gamma$

take unit vecs $v = \frac{1}{\sqrt{3}}(2e_{\alpha_1} + e_{\alpha_2})$, v', v''.

- $(e_{\alpha_0}, v) = \frac{2}{\sqrt{3}}(e_{\alpha_0}, e_{\alpha_1}) = -\frac{1}{\sqrt{3}}$
- v, v', v'' mutually orth, don't span e_{α_0}

$\Rightarrow \underbrace{(e_{\alpha_0}, v)^2}_{\text{but is } \frac{1}{3}} + \underbrace{(e_{\alpha_0}, v')^2}_{\text{same}} + \underbrace{(e_{\alpha_0}, v'')^2}_{\text{same}} < 1$ ☹

Dynkin diagram to Lie alg rels.

$\Pi = \{\alpha_1, \ldots, \alpha_n\}$. enumerate.

Def. Cartan matrix $A = (a_{ij})_{i,j=1}^{n}$, $a_{ij} = n_{\alpha_i, \alpha_j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$

(another conv. $a_{ij} = n_{\alpha_j, \alpha_i}$)

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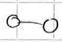
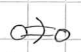
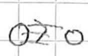
Ex. $A_2 \leftrightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $B_2 \leftrightarrow \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$, $G_2 \leftrightarrow \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$





1 2



1 2

Goal: construct $\left(\begin{array}{l} \text{simple Lie alg } \mathfrak{g}_\Gamma \\ \text{Cartan subalg } \mathfrak{h}_\Gamma \subset \mathfrak{g}_\Gamma \end{array} \right.$

with root set $R = R^+ \sqcup R^-$ s.t.

simple roots $\equiv$ labeled by $1, \dots, n$.

- Π from $(\mathfrak{g}_\Gamma, \mathfrak{h}_\Gamma) \equiv$ vertices of Γ
- inn. prod. $\alpha', \beta \in \Pi$ w.r.t. Killing form

  $=$ corresponding angle in (E_Γ, R_Γ)

  (given by # edges in Γ)

How to construct $(\mathfrak{g}_\Gamma, \mathfrak{h}_\Gamma)$:

mimick relations of $\underset{\mathfrak{g}_\alpha}{\underset{\Uparrow}{E_\alpha}}$, $\underset{\mathfrak{g}_{-\alpha}}{\underset{\Uparrow}{F_\alpha}}$, $H_\alpha = [E_\alpha, F_\alpha]$

coming from simple Lie alg $\mathfrak{g}$, Cartan $\mathfrak{h}$.

write rels. in terms of $\dfrac{2(\alpha, \beta)}{(\alpha, \alpha)} = \beta(H_\alpha)$ - (★)

(10.23.)

"Obivious" relations

R1) $[E_\alpha, F_\alpha] = H_\alpha$ (def. of H_α)

R2) $[E_\alpha, F_\beta] = 0$ (will be in $\mathfrak{g}_{\alpha - \beta} = 0$)

$\alpha - \beta$ not root (11.05)

R3) $[H_\alpha, H_\beta] = 0$ $\mathfrak{h}$ comm.

R4) $[H_{\alpha_i}, E_{\alpha_j}] \underset{\substack{\uparrow \\ E_{\alpha_j} \in \mathfrak{g}_{\alpha_j}}}{=} \alpha_j(H_{\alpha_i}) E_{\alpha_j} \underset{(★)}{=} \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} E_{\alpha_j} = a_{ij} E_{\alpha_j}$

$[H_{\alpha_i}, F_{\alpha_j}] = -a_{ij} F_{\alpha_j}$

Not-so-obvious rel (Serre relations)

R5) $\mathrm{Ad}_{E_{\alpha_i}}^{1-a_{ij}}(E_{\alpha_j}) = 0 = \mathrm{Ad}_{F_{\alpha_i}}^{1-a_{ij}}(E_{\alpha_j})$

Ex. for $\mathfrak{sl}_3(\mathbb{C})$, $\alpha_1 = \alpha_{12}$, $\alpha_2 = \alpha_{23}$ $a_{12} = -1$.

$\rightsquigarrow [E_{12}, [E_{12}, E_{23}]] = [E_{12}, E_{13}] = 0$

Proof of Serre rel. (for E's)

The α_i-string through α_j

$\alpha_j - p\alpha_i, \alpha_j - (p-1)\alpha_i, \dots, \alpha_j, \alpha_j + \alpha_i, \dots, \alpha_j + q\alpha_i$

Step 1 $p = 0$: α_i, α_j simple $\Rightarrow \alpha_j - \alpha_i$ not root

Step 2 $q = -\dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = -a_{ij}$

$\because$ Lem in 11.06 $(p - q = n_{\alpha_i, \alpha_j})$

& Step 1.

So $\alpha_j + (q+1)\alpha_i$ not root

$\mathrm{Ad}_{E_{\alpha_i}}^{1-a_{ij}}(E_{\alpha_j}) \in \mathfrak{g}_{\alpha_j + (q+1)\alpha_i} = 0$ $\square$

So : starting from Dynkin diagram

(or Cartan matrix $A = (a_{ij})_{i,j=1}^n$)

$\rightsquigarrow$ Consider Lie alg $\mathfrak{g} = \mathfrak{g}_n$

generators $E_{\alpha_i}, F_{\alpha_i}, H_{\alpha_i}$ $i = 1, \dots, n$

relations R1 ~ R5

And check $\left(\begin{array}{l} \cdot\ \mathfrak{g} \text{ is simple.} \\ \quad \nexists \text{ no extra relation.} \\ \cdot\ \mathfrak{h} = \langle H_{\alpha_i} : i = 1, \dots, n \rangle \subset \mathfrak{g} \\ \quad \text{Cartan subalg} \end{array} \right.$