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Summary

- free Lie algebras.
- ideals from Serre relation.
- (semi) simplicity of quotient.

Free Lie algebras.

Want: make sense of Lie a'g with generators $x_1, x_2, \cdots$, <u>no relations</u>

cf. - free commutative algebra: polynomial a'g $\mathbb{C}[x_1, x_2, \cdots]$

- free associative algebra: noncomm. polynom. a'g $\mathbb{C}\langle x_1, x_2, \cdots\rangle$

(distinguish $x_1 x_2$ & $x_2 x_1$, etc.)

formally: $V = \{$ linear. comb. of $x_1, x_2, \cdots \}$

$TV = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus \cdots$

with concatenation prod.

$(v_1 \otimes \cdots \otimes v_m) \cdot (u_1 \otimes \cdots \otimes u_n) = v_1 \otimes \cdots \otimes v_m \otimes u_1 \otimes \cdots \otimes u_n$

Put $U = \mathbb{C}\langle x_1, x_2, \cdots \rangle$, consider hom.

$\Delta : U \to U \otimes U$ s.t. $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$

$$(\Rightarrow \Delta(x_1 x_2) = (x_1 \otimes 1 + 1 \otimes x_1)(x_2 \otimes 1 + 1 \otimes x_2)$$
$$= x_1 x_2 \otimes 1 + x_1 \otimes x_2 + x_2 \otimes x_1 + 1 \otimes x_1 x_2 \quad \cdots (*)$$

Def. Free Lie a'g generated by $x_1, x_2, \cdots$ is $\mathrm{Lie}\langle x_1, x_2, \cdots \rangle = \{ a \in U : \Delta(a) = a \otimes 1 + 1 \otimes a \}$

primitive elements

bracket: $[a, b] = ab - ba$ (in U)

Rem. a, b prim. $\Rightarrow [a, b]$ prim. from $(*)$

To consider Lie a's. w/ gens & rels
$x_1, x_2, \cdots$ $[x_a, x_b] = x_c$, etc.

take quot. $\mathrm{Lie}\langle x_1, x_2, \cdots \rangle / ($ideal generated by relations $[x_a, x_b] - x_c$, etc.$)$

Ideals from Serre relation

(E, R) root system, $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ simple pos. rts

$A = (a_{ij})_{i,j=1}^n$ $\quad a_{ij} = n_{\alpha_i, \alpha_j} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ Cartan mat

Consider Lie alg $\tilde{\mathfrak{g}}$ with

- generators e_i, f_i, h_i $(i = 1, \cdots, n)$
- relations R1 ~ R4 (11.12.)

"all rels other than Serre rel"

$\tilde{\mathfrak{n}}_+$ = suba's gen'ed by $e_i < \tilde{\mathfrak{g}}$, $\tilde{\mathfrak{n}}_-$ -- by f_i

Rem. $\theta(e_i) = -f_i$, $\theta(f_i) = -e_i$, $\theta(h_i) = -h_i$
is an aut. of $\tilde{\mathfrak{g}}$ ("$\theta(X) = -X^t$")

Put $x_{ij} = \mathrm{ad}_{e_i}^{1-a_{ij}}(e_j)$ $\quad \tilde{\mathfrak{n}}_+^0$: ideal of $\tilde{\mathfrak{n}}_+$
generated by $(x_{ij})_{i,j=1}^n$

$y_{ij} = \mathrm{ad}_{f_i}^{1-a_{ij}}(e_j)$ $\quad \tilde{\mathfrak{n}}_-^0$ ---

Proposition $[\tilde{\mathfrak{n}}_-, x_{ij}] = 0$ $\quad ([\tilde{\mathfrak{n}}_+, y_{ij}] = 0)$

Cor. $\tilde{\mathfrak{n}}_\pm^0 \triangleleft \tilde{\mathfrak{g}}$

$\because$ - h_i stabilizes $\tilde{\mathfrak{n}}_+^0$
x, x' eigenvec for ad_{h_i}
$\Rightarrow$ so is $[x, x']$.

- e_i stabilizes $\tilde{\mathfrak{n}}_+^0$ by def
- f_i stabilizes $\tilde{\mathfrak{n}}_+^0$ by Prop.

Proof of Prop.

Step 1. $[f_k, x_{ij}] = 0$ for $k \neq i, j$

$\because$ $[f_k, e_i] = 0 = [f_k, e_j]$ (R2)

Step 2 $[f_j, x_{ij}] = 0$

$\because$ f_j comm. w/ e_i $\Rightarrow$ ad_{f_j} comm. w/ ad_{e_i}

$\Rightarrow$ $[f_j, x_{ij}] = \mathrm{ad}_{e_i}^{1-a_{ij}}(\underbrace{[f_j, e_j]}_{-h_j}) = \mathrm{ad}_{e_i}^{-a_{ij}}(\underbrace{-[e_i, h_j]}_{a_{ji} e_i \text{ by (R4)}})$

If $a_{ji} = 0$ this is zero.

if $a_{ji} \neq 0$ a_{ij} is also nonzero.

$\rightsquigarrow$ we have $\mathrm{ad}_{e_i}^{-a_{ij}}(e_i) = 0$

Step 3 $[f_i, x_{ij}] = 0$

$\because$ From $[\mathrm{ad}_{f_i}, \mathrm{ad}_{e_i}] = \mathrm{ad}_{[f_i, e_i]} \underset{(R1)}{=} -\mathrm{ad}_{h_i}$

and $[\mathrm{ad}_{h_i}, \mathrm{ad}_{e_i}] = \mathrm{ad}_{[h_i, e_i]} \underset{(R4)}{=} 2\,\mathrm{ad}_{e_i}$

$[\mathrm{ad}_{f_i}, \mathrm{ad}_{e_i}^k] = -k\,\mathrm{ad}_{e_i}^{k-1}(\mathrm{ad}_{h_i} + k - 1)$

by induction

Apply this for $k = 1 - a_{ij}$, use

$\mathrm{ad}_{f_i}(e_j) = 0$, etc. $\blacksquare$

Goal: $\mathfrak{g}_A = \tilde{\mathfrak{g}} / (\tilde{\mathfrak{n}}_+^0 + \tilde{\mathfrak{n}}_-^0)$ is (semi) simple.

E_i, F_i, H_i imgs of e_i, f_i, h_i

- Implementing Weyl group action.

Want: impl $W \curvearrowright \mathfrak{h}$ as "adjoint by elements of $\exp(\mathfrak{g}_A)$"

Def. <u>Derivation</u> is $D: \mathfrak{g} \to \mathfrak{g}$ s.t.

$D([x, y]) = [D(x), y] + [x, D(y)]$

Ex. $D(x) = [z, x]$ for fixed $z \in \mathfrak{g}$.

D is <u>locally nilpotent</u> if $\forall x \in \mathfrak{g}\ \exists k\ D^k(x) = 0$

$\rightarrow e^D(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k(x)$ makes sense in $\mathfrak{g}$

$e^D : \mathfrak{g} \to \mathfrak{g}$ Lie alg aut

For $\mathfrak{g} = \mathfrak{g}_A$, ad_{E_i}, ad_{F_i} are (loc) nilpot.

e.g. $ad_{E_i}^k(E_j) = 0$, $k = 1 - a_{ij}$; $ad_{E_i}^k(H_j) = 0$, $k = 2$; $ad_{E_i}^k(F_j) = 0$, $k = 1$, etc.

Put. $\theta_i = e^{ad_{E_i}} e^{-ad_{F_i}} e^{ad_{E_i}} \in \mathrm{Aut}(\mathfrak{g}_A)$

Motivation for $\mathfrak{sl}_2$, $e^{ad_E} e^{-ad_F} e^{ad_E} = Ad_{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$

Lem $\theta_i(H_j) = H_j - a_{ij} H_i$ i.e. $\theta_i \leftrightarrow s_{\alpha_i}$

· Semisimplicity of $\mathfrak{g}_A$

Thm (E, R) (irred.) root system

$\Rightarrow \mathfrak{g}_A$ (semi) simple

Proof. Step 1 Enough to prove $\nexists$ nonzero commutative ideal $\triangleleft\, \mathfrak{g}_A$

$\because$ $\mathrm{Rad}\,\mathfrak{g} \neq 0 \Rightarrow D^k(\mathrm{Rad}(\mathfrak{g}))$ comm. at some k

Step 2 $\mathfrak{b} \triangleleft \mathfrak{g} \Rightarrow \mathfrak{b} = (\mathfrak{b} \cap \mathfrak{h}_0) \oplus (\bigoplus_{\alpha:\text{roots}} \mathfrak{b} \cap \mathfrak{g}_\alpha)$

$\mathfrak{h}_0 = \langle H_i : i = 1, \dots, n \rangle$

$\because$ ad_{H_i} mutually comm, diagonalized on $\mathfrak{g}_A$

& $\mathfrak{g} = \mathfrak{h}_0 \oplus (\bigoplus_{\alpha\ \text{roots}} \mathfrak{g}_\alpha)$ fact

Step 3 $\mathfrak{b} \cap \mathfrak{h}_0 \neq 0 \Rightarrow \exists i\ E_i \in \mathfrak{b}$ ($\Rightarrow \mathfrak{b}$ noncomm.) $F_i = \theta_i(E_i) \in \mathfrak{b}$.

$\because$ For simplicity suppose (E, R) irred.

$\Rightarrow W \curvearrowright \mathfrak{h}_0$ is irred.

W acts by inner aut $\Rightarrow \mathfrak{b} \cap \mathfrak{h}_0$ is W-inv.

Step 4. $\mathfrak{b} \cap \mathfrak{g}_\alpha \neq 0 \Rightarrow \mathfrak{b}$ noncomm.

$\because$ $\exists$ w $(= s_\alpha) \in W$ s.t. $w(\alpha) = -\alpha$.

so $w(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$

w is acting by prod. of $(\Theta_i)_{i=1}^n$

$\rightsquigarrow$ inner, preserves $\mathfrak{b}$.

$\Rightarrow$ $\mathfrak{b}$ contains $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ (1-Dim)

$\Rightarrow$ $\mathfrak{b}$ is noncomm. $\boxtimes$

$\rightsquigarrow$ $e^D(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k(x)$ makes sense in $\mathfrak{g}$.

$e^D : \mathfrak{g} \to \mathfrak{g}$ Lie alg aut.

For $\mathfrak{g} = \mathfrak{g}_A$ from above ad_{E_i},