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Summary

- Compact form. $SU(n) < SL_n(\mathbb{C})$ . ..
- Flag manifold $SU(n)/\text{diag}$.

$\simeq SL(n,\mathbb{C})/(\text{upper triang.})$

- Compact form.

Motivation. $SU(2)$ has Lie a'g

$su_2 = \langle x_1, x_2, x_3 \ ; \ [x_i, x_{i+1}] = \frac{1}{2} x_{i+2} \rangle$.

$\frac{1}{2}\begin{bmatrix} i & \\ & -i \end{bmatrix}, \frac{1}{2}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \frac{1}{2}\begin{bmatrix} & i \\ i & \end{bmatrix}$

complexification $su_2 \otimes_{\mathbb{R}} \mathbb{C}$ is $\mathfrak{sl}_2(\mathbb{C})$.

cplx simple.

→ Want : analogue for other simple Lie a'gs?

e.g. $SU(n) \longleftrightarrow SL_n(\mathbb{C})$. $SO(n) \longleftrightarrow so_n(\mathbb{C})$.

$\mathfrak{g}$ (simple) Lie a'g, cplx, ($\mathfrak{h} < \mathfrak{g}$ Cartan subalg)

Def. Real form of $\mathfrak{g}$: real Lie subalg

$\mathfrak{g}_0 < \mathfrak{g}$ s.t. $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}$

G : (cplx) Lie group s.t. $T_eG \simeq \mathfrak{g}$

G_0 : real Lie subgroup s.t. $T_eG \simeq \mathfrak{g}_0$

Prop. TFAE 1) G_0 is cpt.

2) Killing form B is neg. def. on $\mathfrak{g}_0$

$B(X,X) \le 0$, $B(X,Y) \in \mathbb{R}$ for $X, Y \in \mathfrak{g}_0$.

Proof 1) ⇒ 2) Step 1 $B(X,Y) \in \mathbb{R}$.

∵ $\mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}_X \mathrm{ad}_Y) = \mathrm{Tr}_{\mathfrak{g}_0}(\mathrm{ad}_X \mathrm{ad}_Y)$

from basis of $\mathfrak{g}_0$ ⇝ basis of $\mathfrak{g}$.

Step 2 inv. inn. prod. on $\mathfrak{g}_0$

Take a basis $(X_i)_{i=1}^n$ s.t. $(X_i, X_j) = \delta_{ij}$

Invariance $([X,Y],Z) + (Y,[X,Z]) = 0$

$\Rightarrow$ Mat rep. $(a_{ij})_{i,j=1}^n$ of ad_X w.r.t. $(X_i)_i$ is skew symm.

$\because$ $\mathrm{ad}_X(X_j) = \sum_i a_{ij} X_i \Leftrightarrow a_{ij} = ([X, X_j], X_i)$

inv. $\rightsquigarrow$ eq. to $-(X_j, [X, X_i]) = -a_{ji}$

Step 3 $B(X,X) \leq 0$. (will be $-(X,X)$)

$\because$ $B(X,X) = \mathrm{Tr}_{\mathfrak{g}_0}(\mathrm{ad}_X^2) = \sum a_{ij} a_{ji} \underset{\text{St. 2}}{=} \sum -a_{ij}^2$

2) $\Rightarrow$ 1) Prove claim for $\mathrm{Ad}(G_0) < GL(\mathfrak{g}_0)$

$(X,Y) = -B(X,Y)$ is invar Euclidean inn. prod. on $\mathfrak{g}_0$

$\mathrm{Ad}(G_0)$ is a closed subgroup of $\underbrace{SO(n)}_{\text{cpt}}$ ($n = \dim \mathfrak{g}_0$) $\boxed{E}$

How to construct a compact form.

$\mathfrak{h} < \mathfrak{g}$ Cartan subalg. $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha)$

fix pos. rt $R^+ \subset R$, take

$E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in R^+$

s.t. $(E_\alpha, F_\alpha, H_\alpha = [E_\alpha, F_\alpha])$ are like (E, F, H)

(i.e. normalize F_α so $\alpha(H_\alpha) = 2$)

$\mathfrak{g}_0 = \langle \sqrt{-1} H_\alpha, F_\alpha - E_\alpha, \sqrt{-1}(E_\alpha + F_\alpha) : \alpha \in R^+ \rangle$

is a cpt real form. $\rightarrow$ ex. $\mathfrak{su}_n$: Skew Herm.

What's good about cpt form.

ad_X is skew symm. for an(y) ONB.

$\rightsquigarrow$ diagonalizable.

$\Rightarrow$ any max. comm. subalg $\mathfrak{h}_0$ gives Cartan.

Cartan subalg $\mathfrak{h}_0 < \mathfrak{g}_0$

$\Leftrightarrow$ maximal comm subgrp $T < G_0$

(<u>maximal</u> <u>torus</u>)

Flag mfd.

$T < G_0$, $H < G$ subgrps for Cartan subalgs.

$$\mathfrak{b} = \underbrace{\mathfrak{h} \oplus \underbrace{\left(\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha\right)}_{\text{nilpot ideal of } \mathfrak{b}}}_{\text{solvable}}$$ "upper triangular" subalg

$B < G$ corresp. subgroup (Borel subgroup)

Prop. $G/B = G_0/T$.

Proof. Show $G_0 \curvearrowright G/B$ is transitive, 1)

2) $\{ g \in G_0 : g[B] = [B] \}$ is T.
stabilizer

1) Want: $\dim G_0[B] = \dim_{\mathbb{R}} G/B = \dim_{\mathbb{R}} \mathfrak{g} \ominus \mathfrak{b}$.)
$= 2 \cdot \#(R^-)$

(2) $\mathfrak{b} \cap \mathfrak{g}_0 = \langle \sqrt{-1} H_\alpha \rangle = \mathfrak{h}_0$

$\dim G_0[B] = \dim(\mathfrak{g}_0 \ominus \mathfrak{h}_0) = 2 \#(R^+)$.

$\#(R^-) = \#(R^+)$ by $\alpha \leftrightarrow -\alpha$

Ex. $G = SL_n(\mathbb{C})$ B: upper triangular mats

$$G/B \simeq \{ \underbrace{(V_1, \ldots, V_{n-1})}_{\text{flag in } \mathbb{C}^n} : V_i \subset V_{i+1} \subset \mathbb{C}^n \text{ subsp.}, \ \dim V_k = k \}$$

$g(V_1, \ldots, V_n) = (gV_1, \ldots, gV_n)$

B: stabilizer of $(V_i)_i$ w/ $V_k = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} : z_j \in \mathbb{C} \right\} \subset \mathbb{C}^n$

Bruhat decomposition $\quad W$: Weyl group

$$G = \coprod_{w \in W} B \hat{w} B \qquad \hat{w} \in G \text{ s.t. } \mathrm{Ad}_{\hat{w}} = w.$$

i.e. double coset decomp for $B < G$ is

$$B \backslash G / B = \{ [\hat{w}] : w \in W \}$$

$\rightsquigarrow$ $G/B = \coprod_{w \in W} B \hat{w} B / B$.

Rem. $N_- \leq G$ subgroup corr. to $\mathfrak{n}_- = \bigoplus_{\alpha \in R_-} \mathfrak{g}_\alpha$ (nilpot, complex).

$B \hat{w} B / B$: N_--orbits of G/B.

$\rightsquigarrow$ isom to $\mathbb{C}^{\ell(w)}$ for some $\ell(w) \in \mathbb{N}$.

$\rightsquigarrow$ $G/B = G_0/T$ is made up by "glueing" $\mathbb{C}^{\ell(w)}$ for $w \in W$.

Ex. $SL_2(\mathbb{C})/B$ $\left(\simeq \underset{S^3}{SU(2)} / \underset{U(1)}{\text{diag.}} \right) \simeq S^2$

is $\mathbb{C}^0 = pt \amalg \mathbb{C} \simeq \mathbb{R}^2$

Application : any two Cartan subalgs are conjugate.

Outline $\exists X \in \mathfrak{h}$ "generic" $(\alpha(H) \neq 0 \ \forall \alpha \in R)$

s.t. $\mathfrak{h} = \{ Y \in \mathfrak{g} : [Y, X] = 0 \}$ centralizer

$\mathfrak{h}'$, X' another such $\rightsquigarrow$ want to find $g \in G$

s.t. $\mathrm{Ad}_g(X') \in \mathfrak{h}$

Step 1. $\exists g$ $\mathrm{Ad}_g(X') \in \mathfrak{b}$.

$\because$ $e^{X'}$ has fixed pt $g^{-1}B$ in G/B.

by Lefschetz fixed pt formula.

$$\#\text{ fixed pt} \geq \mathrm{Tr}\, e^{X'}|_{H^{ev}(G/B)} - \mathrm{Tr}\, e^{X'}|_{H^{odd}(G/B)}$$

$$\hphantom{\#\text{ fixed pt}} \underset{\hphantom{x}}{} \quad \overset{\hphantom{x}}{} 0$$

Step 2 $\mathrm{Ad}_g(X') \in \mathfrak{b} \Rightarrow \mathrm{Ad}_g(X') \in \mathfrak{h}$ by ss-nilp dec.