Compulsory assignment in MAT4270, Fall 2025

Problem 1. Let G be a (possibly infinite) group. A function $\phi\colon G \to \mathbb{C}$ is called *positive definite* if for any $n \geq 1$, $g_1, \ldots, g_n \in G$ and $z_1, \ldots, z_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{n} \bar{z}_i z_j \phi(g_i^{-1} g_j) \geq 0.$$

(In other words, the matrix $(\phi(g_i^{-1}g_j))_{i,j}$ is positive semidefinite.)

(a) Show that if $\phi\colon G \to \mathbb{C}$ is a positive definite function, then

$$\phi(e) \geq 0, \quad \phi(g^{-1}) = \overline{\phi(g)}, \quad |\phi(g)| \leq \phi(e) \quad (g \in G).$$

(b) Show that if $\pi\colon G \to U(H)$ is a unitary representation of G on a Hilbert space H and $\xi \in H$, then the matrix coefficient $\phi(g) = (\pi(g)\xi, \xi)$ is positive definite.

(c) Prove the converse to (b): for any positive definite function ϕ on G there are a unitary representation π of G on a Hilbert space H and a vector $\xi \in H$ such that $\phi(g) = (\pi(g)\xi, \xi)$ for all $g \in G$. Furthermore, show that the triple (H, π, ξ) is unique up to unitary equivalence if we require in addition the vector ξ to be cyclic, that is, the span of $\pi(G)\xi$ to be dense in H. Hint: for the existence part, define a pre-inner product on $\mathbb{C}G$ by

$$(g, h) = \phi(h^{-1}g).$$

Let us denote by π_ϕ the representation we get by the procedure indicated above.

Problem 2. Let G be a finite group. Consider a finite dimensional representation $\pi\colon G \to \mathrm{GL}(V)$ and its character $\chi = \chi_\pi$.

(a) Show that χ is positive definite.

(b) Assume π is irreducible and take a basis $e_1, \ldots, e_n$ in V. Show that the vector $(e_1, \ldots, e_n)$ is cyclic for the representation $n\pi$. Conclude that $\pi_\chi \sim n\pi$.

(c) More generally, show that if $\pi \sim m_1\pi_1 \oplus \cdots \oplus m_k\pi_k$ for some $m_i \geq 1$ and mutually inequivalent irreducible representations π_i, then

$$\pi_\chi \sim (\dim \pi_1)\pi_1 \oplus \cdots \oplus (\dim \pi_k)\pi_k.$$

Problem 3. Let G be a finite group. Define an involution on $\mathbb{C}G$ by $g^* = g^{-1}$. Thus, the map $\mathbb{C}G \to \mathbb{C}G$, $a \mapsto a^*$, is anti-linear, involutive ($(a^*)^* = a$) and anti-multiplicative ($(ab)^* = b^*a^*$). Any function $\phi\colon G \to \mathbb{C}$ extends by linearity to $\mathbb{C}G$, we continue to denote this extension by ϕ.

(a) Show that a function $\phi\colon G \to \mathbb{C}$ is positive definite if and only if the corresponding linear functional is *positive* in the sense that $\phi(a^*a) \geq 0$ for all $a \in \mathbb{C}G$.

(b) Show that a function $\phi\colon G \to \mathbb{C}$ is central if and only if the corresponding linear functional is a *trace* in the sense that $\phi(ab) = \phi(ba)$ for all $a, b \in \mathbb{C}G$.

(c) Show that the usual matricial trace is the only trace on $n \times n$ matrices up to a scalar factor. Conclude that any positive trace on $\mathbb{C}G$ has the form $\sum_{[\pi]\in\widehat{G}} c_\pi \chi_\pi$ for some $c_\pi \geq 0$.

(d) Prove that a nonzero function $\phi\colon G \to \mathbb{C}$ is the character of an irreducible representation if and only if the following conditions are satisfied:

(1) ϕ is central;
(2) ϕ is positive definite;
(3) $\phi(e)^{-1}\phi$ is an extremal point of the convex set of central positive definite functions $\psi\colon G \to \mathbb{C}$ such that $\psi(e) = 1$;
(4) $\sum_{g \in G} |\phi(g)|^2 = |G|$.