

MAT4301 – Partial differential equations

Mandatory assignment 1 of 1

Submission deadline

Thursday 17 November 2022 at 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between writing in English or Norwegian.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with $\text{\LaTeX}$). The assignment must be submitted as a single PDF. Scanned pages must be clearly legible; please use either the “Color” or “Grayscale” settings. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. As a rule of thumb, you need to correctly answer $2/3$ of the assignment in order to get it passed, and to show that you have made an attempt at solving all of the problems.

All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Definition. Let $n, m \in \mathbb{N}$ and let $U \subset \mathbb{R}^n$. We define the L^2 -norm of a function $v: U \rightarrow \mathbb{R}^m$ as the number

$$\|v\|_{L^2(U)} = \left(\int_U |v(x)|^2 dx \right)^{1/2}.$$

The set $L^2(U, \mathbb{R}^m)$ is the set of all functions from U to $\mathbb{R}^m$ whose L^2 -norm is finite. When $m = 1$ we write $L^2(U) = L^2(U, \mathbb{R})$.

Remark. The Cauchy–Schwarz inequality says that if $u, v \in L^2(U)$ then $\int_U uv \, dx \leq \|u\|_{L^2(U)} \|v\|_{L^2(U)}$.

Problems

1. Let $u \in C^2(\mathbb{R}^n)$ solve the Poisson equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^n \quad (1)$$

where $f \in C(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Assume that u and its derivatives lie in $L^2(\mathbb{R}^n)$. (In particular, u and its derivatives go to 0 at infinity.) Prove that

$$\|u_{x_i x_j}\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} \quad \text{for all } i, j = 1, \dots, n. \quad (2)$$

Remark. Clearly, (1) implies that $\|u_{x_1 x_1} + \dots + u_{x_n x_n}\|_{L^2(U)} = \|f\|_{L^2(U)}$. What is remarkable is that (1) implies that all mixed derivatives are automatically bounded.

2. (a) The Poincaré inequality states that for every bounded domain $U \subset \mathbb{R}^n$ there exists a constant $C > 0$ such that

$$\|u\|_{L^2(U)} \leq C \|Du\|_{L^2(U)} \quad (3)$$

for every $u \in C^1(U) \cap C(\bar{U})$ with $u|_{\partial U} \equiv 0$. Prove the Poincaré inequality in $n = 1$ dimensions for intervals $U = [a, b]$.

Hint. First prove the identities

$$\frac{d}{dx} u(x)^2 = 2u(x)u'(x) \quad \Rightarrow \quad u(x)^2 = 2 \int_a^x u(y)u'(y) \, dy.$$

Then integrate over $x \in [a, b]$ and apply the Cauchy–Schwarz inequality.

- (b) Prove the Gronwall inequality: If $\alpha \in C^1([0, \infty))$ satisfies

$$\alpha'(t) \leq b(t)\alpha(t) \quad \forall t > 0 \quad (4)$$

for some $b \in C([0, \infty))$ then

$$\alpha(t) \leq \alpha(0)e^{B(t)} \quad (5)$$

for all $t \geq 0$, where $B(t) = \int_0^t b(s) \, ds$.

Hint. Multiply (4) by an integrating factor.

3. (Long-term behavior of the heat equation) Let $U \subset \mathbb{R}^n$ be an open, bounded domain. Consider the initial-boundary value problem

$$\begin{cases} u_t = \Delta u + f & \text{in } U \times (0, \infty) \\ u = g & \text{on } \partial U \times [0, \infty) \\ u = h & \text{on } U \times \{t = 0\} \end{cases} \quad (6)$$

where the source term f , boundary data g and initial data h are given and $f \in C(\bar{U} \times [0, \infty))$, $g \in C(\partial U \times [0, \infty))$ and $h \in C(\bar{U})$.

- (a) Let $u, v \in C_1^2(U \times (0, \infty)) \cap C(\bar{U} \times [0, \infty))$ both satisfy (6) with the same functions f and g , but with different initial data h_1 and h_2 , respectively. Prove the *energy estimate*

$$\frac{d}{dt} E[u - v](t) = - \int_U |D(u - v)(x, t)|^2 dx \quad (7)$$

where $E[w](t) = \frac{1}{2} \int_U |w(x, t)|^2 dx$.

- (b) Prove that there is a constant $C > 0$ such that

$$E[u - v](t) \leq \frac{1}{2} \|h_1 - h_2\|_{L^2(U)}^2 e^{-Ct}. \quad (8)$$

Hint. Use (7) and the Poincaré and Gronwall inequalities.

- (c) Conclude that the difference $|u(x, t) - v(x, t)|$ converges to 0 as $t \rightarrow \infty$, no matter how h_1, h_2 were chosen.

Remark. We can not conclude from the above that u and v converge to some common limit as $t \rightarrow \infty$, only that they “become more and more similar”.

- (d) Assume now that f and g are constant in time, i.e. $f(x, t) = \bar{f}(x)$ for all $x \in U, t > 0$ and $g(x, t) = \bar{g}(x)$ for all $x \in \partial U, t > 0$, where $\bar{f} \in C(\bar{U})$ and $\bar{g} \in C(\partial U)$. Let u solve (6) as before, and let $\bar{u} \in C^2(U) \cap C(\bar{U})$ solve the Poisson problem

$$\begin{cases} -\Delta \bar{u} = \bar{f} & \text{in } U \\ \bar{u} = \bar{g} & \text{on } \partial U. \end{cases} \quad (9)$$

Use (b) to show that $u(\cdot, t) \rightarrow \bar{u}$ in $L^2(U)$ as $t \rightarrow \infty$. How fast does this convergence take place?

4. Let $U \subset \mathbb{R}^n$ be open, bounded and connected. We say that $u \in C^2(U)$ is *subharmonic* if

$$\Delta u \geq 0 \quad \text{in } U. \quad (10)$$

- (a) Assume that $u \in C^2(U)$ is subharmonic. *Sketch* a proof of the fact that

$$u(x) \leq \int_{B(x, r)} u(y) dy \quad (11)$$

for all $x \in U$ and $r > 0$ such that $B(x, r) \subset U$.

- (b) Prove the strong maximum principle for subharmonic functions: If $u \in C^2(U) \cap C(\overline{U})$ is subharmonic and there is some $x_0 \in U$ such that $u(x_0) = \max_{\overline{U}} u$, then u is constant.
- (c) Let $u \in C^2(U) \cap C(\overline{U})$ satisfy the PDE

$$\begin{cases} \Delta u = f(u) & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases} \quad (12)$$

for some given (possibly nonlinear) function $f \in C(\mathbb{R})$.

- (i) Show that if $f \geq 0$ then either $u \equiv 0$, or $u < 0$ in U .
- (ii) Show that if $f(u) = u(u^2 - 1)$ then $-1 \leq u(x) \leq 1$ for all $x \in U$.

Hint. Assume that there is some $x_0 \in U$ where $u(x_0) = \min_{\overline{U}} u < -1$. Show that this is in contradiction with (12).