

Problem sheet 5

MAT4301

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1. (Part of the proof of Theorem 6) Let $T, \varepsilon > 0$. Show that the function

$$w(x, t) = \frac{1}{(T + \varepsilon - t)^{n/2}} e^{|x|^2/4(T+\varepsilon-t)}$$

satisfies the heat equation $w_t = \Delta w$ for $t \in (0, T]$, $x \in \mathbb{R}^n$. Note that $w(x, t)$ increases very quickly as $|x| \rightarrow \infty$.

2. (a) (Problem 2.5.15 from Evans) Let $g: [0, \infty) \rightarrow \mathbb{R}$ satisfy $g(0) = 0$. Derive the formula

$$u(x, t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-x^2/(4(t-s))} g(s) ds \quad (1)$$

for a solution of the one-dimensional initial/boundary value problem

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u = 0 & \text{on } \mathbb{R}_+ \times \{0\} \\ u = g & \text{on } \{0\} \times [0, \infty). \end{cases} \quad (2)$$

(Here, $\mathbb{R}_+ := (0, \infty)$.)

Hint: Define $v(x, t) := u(x, t) - g(t)$ and extend v to $\mathbb{R} \times [0, \infty)$ by the odd reflection $v(x, t) = -v(-x, t)$ for $x < 0$.

- (b) Find a solution formula for the full problem

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u = h & \text{on } \mathbb{R}_+ \times \{0\} \\ u = g & \text{on } \{0\} \times [0, \infty). \end{cases} \quad (3)$$

for functions g, h satisfying $g(0) = h(0)$.

3. Consider the Cauchy problem for the heat equation

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R} \times (0, \infty) \\ u(x, 0) = 0 & \text{for } x \in \mathbb{R}. \end{cases} \quad (4)$$

Clearly, the trivial solution $u \equiv 0$ is one solution of (4). Let now $\alpha > 1$ and define

$$v(x, t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}, \quad g(t) := \begin{cases} e^{-1/t^\alpha} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases} \quad (5)$$

where $g^{(k)}$ is the k -th derivative of g . Show – either rigorously or by doing formal computations – that v also solves (4).

4. Consider a *discrete random walk* in one dimension: At each point $x_i = i \Delta x$ and $t^n = n \Delta t$ (where $\Delta x, \Delta t > 0$ are given parameters and $i \in \mathbb{Z}, n \in \mathbb{N}_0$) we have a lump of particles which in the time interval $[t^n, t^{n+1}]$ has a probability p of moving to the left to x_{i-1} and probability p of moving right to x_{i+1} . (Then the probability of staying put at x_i is $1 - 2p$, so we need $p \in [0, 1/2]$.) At time $t = 0$ we start with a distribution of particles $\{u_i^0\}_{i \in \mathbb{Z}} \subset \mathbb{R}$ satisfying $u_i^0 \geq 0$ for all i .

- (a) Let the distribution $\{u_i^n\}_{i \in \mathbb{Z}}$ at time t^n be given. Explain why

$$u_i^{n+1} = pu_{i-1}^n + (1 - 2p)u_i^n + pu_{i+1}^n. \quad (6)$$

- (b) Derive the relation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = k \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \quad (7)$$

where $k := p \frac{\Delta x^2}{\Delta t}$.

- (c) Define $u_{\Delta t, \Delta x}(x_i, t^n) = u_i^n$, and extend $u_{\Delta t, \Delta x}$ to all points $(x, t) \in \mathbb{R} \times [0, \infty)$ by linear interpolation. We now wish to let $\Delta t, \Delta x \rightarrow 0$, but we have to be careful about how fast Δt goes to zero compared to Δx .

Assume that $u_{\Delta t, \Delta x}$ converges to some function u . In the following three limits, find a differential equation satisfied by u :

- i. $\Delta t, \Delta x \rightarrow 0$ such that $k := p \frac{\Delta x^2}{\Delta t} \rightarrow 0$
- ii. $\Delta t, \Delta x \rightarrow 0$ such that $k := p \frac{\Delta x^2}{\Delta t} \equiv \text{const.}$
- iii. $\Delta t, \Delta x \rightarrow 0$ such that $k := p \frac{\Delta x^2}{\Delta t} \rightarrow \infty$.

- (d) Relate the space-time scaling $\frac{\Delta x^2}{\Delta t} \equiv \text{const.}$ to what you know about the symmetries of the heat equation.

Note: Answering the above problems fully rigorously requires a lot of work, so formal explanations are enough.

5. Repeat problem 4, but assume that in addition to particles jumping to neighbouring points, there is a production or destruction of particles. (For instance, the particles could be bacteria, and bacteria could die or reproduce.) More precisely, at each point x_i there is in the time interval $[t^n, t^{n+1}]$ a production $\Delta t f(x_i, t^n)$ of particles, for some function $f: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$.
6. Assume now that there is a local *drift* of particles at speed $b \in \mathbb{R}$. For definiteness, assume $b > 0$.

- (a) Explain why the new probability of moving from x_i to x_{i+1} in the time interval $[t^n, t^{n+1}]$ is $p + \frac{\Delta t}{\Delta x} b$, the probability of moving from x_i to x_{i-1} is p , and the probability of staying put is $1 - 2p - \frac{\Delta t}{\Delta x} b$.
- (b) Repeat problem 4 for particles with drift.