

Problem sheet 6

MAT4301

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1. (a) Write down d'Alembert's formula for the solution of the one-dimensional wave equation with wave speed $c > 0$,

$$\begin{cases} u_{tt} = c^2 u_{xx} & (x \in \mathbb{R}, t > 0) \\ u(x, 0) = g(x) \\ u_t(x, 0) = h(x). \end{cases} \quad (1)$$

Draw the backwards wave cone from a point $(x, t) \in \mathbb{R} \times \mathbb{R}_+$.

Remark. The backwards wave cone at (x, t) is the set of points (y, s) with $s < t$ such that the solution at (x, t) depends on the solution at (y, s) .

- (b) Solve (1) when $c = 1$, $g(x) = \sin(x)$, $h(x) = \cos(2x)$.
2. Consider the one-dimensional wave equation on $\mathbb{R}_+$ with *Neumann boundary condition*,

$$\begin{cases} u_{tt} = c^2 u_{xx} & (x > 0, t > 0) \\ u(x, 0) = g(x) & (x > 0) \\ u_t(x, 0) = h(x) & (x > 0) \\ u_x(0, t) = 0 & (t > 0). \end{cases} \quad (2)$$

Where g, h are given C^1 functions satisfying $g'(0) = h'(0) = 0$.

- (a) Find a solution formula for (2).
Hint: See [p. 69, Evans]. Make an even reflection.
- (b) Make a physical interpretation of the boundary condition and of your solution formula.
3. In this problem we generalize Duhamel's principle to higher-order time-dependent equations. Consider the following second-order linear differential equation: Find $u: [0, \infty) \rightarrow X$ satisfying

$$\begin{cases} u''(t) = L_0 u(t) + L_1 u'(t) + f(t) & (t > 0) \\ u(0) = u_0 \\ u'(0) = u_1 \end{cases} \quad (3)$$

for some given inhomogeneity $f: (0, \infty) \rightarrow X$, initial data $u_0, u_1 \in X$, linear operators $L_0, L_1: X \rightarrow X$, and where X is some vector space (you can put $X = \mathbb{R}$ for simplicity). Assume that you are able to solve the corresponding homogeneous problem

$$\begin{cases} u''(t) = L_0 u(t) + L_1 u'(t) & (t > 0) \\ u(0) = u_0 \\ u'(0) = u_1 \end{cases} \quad (4)$$

and denote the solution by $\varphi(t; u_0, u_1)$ (so that the function $u(t) = \varphi(t; u_0, u_1)$ solves (4)).

(a) Show that u defined by

$$u(t) := \int_0^t \varphi(t-s; 0, f(s)) ds \quad (5)$$

solves the inhomogeneous problem (3) with initial data $u_0 = u_1 = 0$.

(b) Which step of your computation would fail if the operators L_0, L_1 were nonlinear?

(c) Deduce a formula for the solution of the general problem (3).

(d) Generalize Duhamel's principle to the general k -th order linear equation

$$\begin{cases} u^{(k)}(t) = \sum_{j=0}^{k-1} L_j u^{(j)}(t) + f(t) & (t > 0) \\ u^{(j)}(0) = u_j & (j = 0, \dots, k-1) \end{cases} \quad (6)$$

where $u^{(j)}$ (for $j = 0, 1, \dots, k$) are derivatives of u , and L_j are linear operators from X to X .