

Problem sheet 8

MAT4301

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1. Problem 3.5.14 in Evans
2. Problem 10.4.1 in Evans
3. Problem 10.4.2 in Evans (if you have the time)
4. (Problem 10.4.5 in Evans) We say that a bounded, continuous $u: [-1, 1] \rightarrow \mathbb{R}$ is a *viscosity solution* of

$$\begin{cases} |u'(x)| = 1 & \text{in } (-1, 1) \\ u(-1) = u(1) = 0 \end{cases} \quad (1)$$

if $u(-1) = u(1) = 0$, and for all $\varphi \in C^1(\mathbb{R})$ for which $u - \varphi$ has a maximum (or minimum) at $x_0 \in (-1, 1)$, we have $|\varphi'(x_0)| \leq 1$ (or $|\varphi'(x_0)| \geq 1$).

- (a) Show that $u(x) := 1 - |x|$ is a viscosity solution of (1).
- (b) Show that $\tilde{u}(x) := |x| - 1$ is *not* a viscosity solution of (1).
- (c) Show that $\tilde{u}$ is a viscosity solution of

$$\begin{cases} -|u'(x)| = -1 & \text{in } (-1, 1) \\ u(-1) = u(1) = 0. \end{cases} \quad (2)$$

(First specify what we mean by a viscosity solution of (2).)

- (d) Show that u and $\tilde{u}$ solve both (1) and (2), in the sense that the boundary conditions and the PDEs are satisfied at almost every x (i.e., apart from at finitely many points).

Remark. Viewed classically, the two problems (1) and (2) are identical. This problem illustrates that we always need to carefully state what we mean by a “solution”, as different solution concepts might not coincide.

5. Problem 10.4.6 in Evans Let $U \subset \mathbb{R}^n$ be open and bounded. Set $u(x) := \text{dist}(x, \partial U) = \inf_{y \in \partial U} |x - y|$ for $x \in \overline{U}$.
 - (a) Prove that u is Lipschitz continuous.
 - (b) Prove that u is a viscosity solution of the *eikonal equation**

$$\begin{cases} |Du| = 1 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \quad (3)$$

**Eikonal* comes from Greek $\epsilon\iota\kappa\omega\nu$, meaning image or icon. The equation appears, among other places, in the mathematics of optics.

By a viscosity solution, we mean that the boundary data is satisfied pointwise, and whenever $\varphi \in C^\infty(U)$ is such that $u - \varphi$ has a local maximum (or a minimum) at some $x_0 \in U$, then $|D\varphi(x_0)| \leq 1$ (or ≥ 1).