For any positive c set $u_c = \begin{cases} u_1 & t<c \\ u_2(t-c) & t \geq c \end{cases}$

then u_c is also a weak solution!

This is not good, we must resolve the situation...

Vanishing viscosity

(V) $\quad u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u_{xx}, \quad \varepsilon > 0 \qquad u^\varepsilon(x,0) = u_0(x)$

Assume for the moment (we shall prove it later) that $u^\varepsilon \in C^\infty(\mathbb{R} \times (\delta, \infty))$ for any $\delta > 0$, and that $u^\varepsilon(x,\delta) \to u_0(x)$ in $L^1(\mathbb{R})$ as $\delta \to 0$.

How can we show that (V) has unique solutions?

Assume that f is Lipschitz continuous, let u & v be solutions and set $w = u - v$. Then

$$w_t + (f(u) - f(v))_x = \varepsilon w_{xx}$$

$$\boxed{w_t + (Aw)_x = \varepsilon w_{xx}} \qquad A = A(x,t) = \frac{f(u) - f(v)}{u - v}$$

Set $\eta(w) = |w| \qquad \eta'(w) = \operatorname{sgn}(w)$ if $w \neq 0$

Multiply with $\eta'(w)$ and use the chain rule

$$\eta(w)_t + A_x \eta(w) + A(w)\eta(w)_x = \eta'(w)\varepsilon w_{xx}$$

$$\eta(w)_t + (A\eta(w))_x = \varepsilon \eta(w)_{xx} - \varepsilon \eta''(w)(w_x)^2$$

← This does not really exist, formal

$$\eta(w)_x + (Aw)_x \leq \varepsilon \eta(w)_{xx}$$

← Can be proved anyway!

Integrate over $\mathbb{R}$ to get

$$\frac{d}{dt}\int_{\mathbb{R}} |w|\,dx \leq 0 \quad \Rightarrow \|w(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|w(\cdot,0)\|_{L^1(\mathbb{R})}.$$

Then (V) has unique solutions.

If we demand that

$$u = \lim_{\varepsilon \to 0} u^\varepsilon$$

maybe the uniqueness property will be inherited by u?

SPOILER: YES

Plan:

1. Prove that $u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}$ has a unique sol.
2. Prove that, with some extra condition, $u_t + f(u)_x = 0$ has unique solutions.
3. Prove that $\lim_{\varepsilon \to 0} u^\varepsilon = u$

$\lim_{\varepsilon \to 0} u^\varepsilon$: solves the viscous problem

u: solves the conservation law

(1) Prelims, Euler's gamma function

$$\Gamma(p) = \int_0^\infty e^{-s} s^{p-1} ds$$

$$\Gamma(p+1) = p\Gamma(p), \quad \Gamma(1) = 1 \Rightarrow \Gamma(n) = (n-1)! \quad n \in \mathbb{N}$$

Euler's beta function

$$B(p,q) = \int_0^1 s^{p-1}(1-s)^{q-1} ds \qquad B(p,q) = B(q,p)$$

Lemma: $$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Proof:

$$\Gamma(p)\Gamma(q) = \int_0^\infty \int_0^\infty e^{-(r+s)} s^{p-1} r^{q-1} ds\, dr$$

$s = \sigma\tau$
$r = \tau(1-\sigma)$
$\sigma \in [0,1]$
$\tau \in [0,\infty)$

$$\frac{\partial(s,r)}{\partial(\sigma,\tau)} = \begin{pmatrix} \tau & \sigma \\ -\tau & 1-\sigma \end{pmatrix}$$

$ds\,dr = \tau\, d\tau\, d\sigma$

$$= \int_0^\infty \int_0^1 e^{-\tau} \sigma^{p-1} \tau^{p-1} \tau^{q-1} (1-\sigma)^{q-1} \tau\, d\sigma\, d\tau$$

$$= \int_0^\infty e^{-\tau} \tau^{p+q-1} d\tau \int_0^1 \sigma^{p-1}(1-\sigma)^{q-1} d\sigma = \Gamma(p+q) B(p,q) \quad \blacksquare$$

$$\Gamma(1/2)^2 = B(1/2,1/2) = \int_0^1 \frac{1}{\sqrt{s(1-s)}} ds = \pi \quad \leftarrow \text{set } s = \sin^2\theta$$

We want to show that the problem:

$$\begin{cases} u_t + f(u)_x = u_{xx} & t > 0 \quad x \in \mathbb{R} \\ u = u_0 & t = 0 \end{cases}$$

has a classical solution.

$t \mapsto t/\varepsilon$, $x \mapsto x/\varepsilon$

Picard iteration

$$\begin{cases} (\partial_t - \partial_{xx})u = -f(u)_x & t>0 \\ u = u_0 & t=0 \end{cases}$$

Formally: $u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\, u_0(y)\, dy - \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s) \frac{\partial}{\partial y} f(u(y,s))\, ds dy$

Now define the sequence $\{u^n\}_{n=0}^{\infty}$ by:

$$u^0 = \int_{\mathbb{R}} \Phi(x-y,t)\, u_0(y)\, dy$$

$$u^{n+1} = u^0 - \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s) \frac{\partial}{\partial y} f(u^n(y,s))\, dy ds \quad n = 1,2,3,\dots$$

$$= u^0 + \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} \Phi(x-y,t-s)\, f(u^n(y,s))\, dy ds.$$

We shall show that $u^n \longrightarrow u$ for some function u.

Notation: $u^{-1} := 0$

$$\boxed{v^n = u^n - u^{n-1}} \qquad n = 0,1,2,3,\dots$$

$$|v^{n+1}(x,t)| = \left| \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} \Phi(x-y,t-s) \left(f(u^n(y,s)) - f(u^{n-1}(y,s)) \right) dy ds \right|$$

$$\leq L \int_0^t \|v^n(\cdot,s)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \left| \frac{\partial}{\partial y} \Phi(y,t-s) \right| dy\, ds \qquad L = \|f\|_{Lip}$$

$y \longmapsto \Phi(y,s)$ is decreasing for $y>0$ and increasing for $y<0$

$$\int_{\mathbb{R}} \left| \frac{\partial}{\partial y} \Phi(y,s) \right| dy = 2 \int_{-\infty}^{0} \frac{\partial}{\partial y} \Phi(y,s) = 2\,\Phi(0,s) = \frac{1}{\sqrt{\pi s}}$$

$$|v^{n+1}(x,t)| \le \frac{L}{\sqrt{\pi}} \int_0^t \|v^n(\cdot,s)\|_{L^\infty(\mathbb{R})} \frac{1}{(t-s)^{1/2}}\, ds$$

$$\Downarrow$$

$$\|v^{n+1}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \frac{L}{\sqrt{\pi}} \int_0^t \|v^n(\cdot,s)\|_{L^\infty(\mathbb{R})} \frac{1}{(t-s)^{1/2}}\, ds$$

Write $V^n(s) = \|v^n(\cdot,s)\|_{L^\infty(\mathbb{R})}$, in this notation:

$$V^{n+1}(t) \le \frac{L}{\sqrt{\pi}} \int_0^t \frac{1}{(t-s)^{1/2}} V^n(s)\, ds$$

Claim:

$$V^n(t) \le M L^n \frac{t^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)} \qquad M \ge \|u_0\|_{L^\infty(\mathbb{R})}$$

Proof: Induction.

$n = 0$

$v^0 = u^0 = \int_{\mathbb{R}} \Phi(x-y,t)\, u_0(y)\, dy$, so that

LHS: $|v^0(x,t)| \le \|u_0\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \Phi(x-y,t)\, dy = \|u_0\|_{L^\infty(\mathbb{R})} \le M$

RHS: $M/\Gamma(1) = M$ ✓

Assume true for n.

$$V^{n+1}(t) \le \frac{ML^{n+1}}{\sqrt{\pi}} \frac{1}{\Gamma\left(\frac{n+2}{2}\right)} \int_0^t s^{n/2} (t-s)^{-1/2}\, ds \qquad \sigma = \frac{s}{t}$$

$$= \frac{ML^{n+1}}{\sqrt{\pi}} \frac{1}{\Gamma\left(\frac{n+2}{2}\right)} \int_0^1 t^{n/2} \sigma^{n/2} t^{-1/2} (1-\sigma)^{1/2} t\, d\sigma$$

$$= \frac{M L^{m+1}}{\sqrt{\pi}} \frac{1}{\Gamma\left(\frac{n+2}{2}\right)} t^{\frac{n+1}{2}} \int_0^1 \sigma^{n/2} (1-\sigma)^{1/2} \, d\sigma$$

$$= \frac{M L^{m+1}}{\sqrt{\pi}} \frac{t^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+2}{2}\right)} B\left(\frac{1}{2}, \frac{n+2}{2}\right)$$

$$= \frac{M L^{m+1}}{\sqrt{\pi}} \frac{t^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+2}{2}\right)} \frac{\Gamma(1/2)\,\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)}$$

$$= M L^{m+1} t^{\frac{n+1}{2}} \frac{1}{\Gamma\left(\frac{n+3}{2}\right)}, \quad \checkmark$$

which finishes the proof of the claim. ∎

For $t < \infty$, $v^n(x,t) \longrightarrow 0$ exponentially fast as $n \to \infty$.

so that

$$u^n = \sum_{k=0}^{n} v_k \longmapsto u \quad \text{in the sup norm.}$$

Furtherfore u satisfies:

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\, u_0(y)\, dy - \int_0^t \int_{\mathbb{R}} \Phi(x-y, t-s) \frac{\partial}{\partial y} f(u(y,s))\, dy\, ds$$

Can show that this implies that u_x, u_{xx} are bounded

$\Rightarrow$ $u \in C^\infty(\mathbb{R} \times (0,\infty))$, and u solves

$$\begin{cases} u_t + f(u)_x = u_{xx} & t > 0 \\ u = u_0 & t = 0 \end{cases}$$

A priori bounds on u^ε

"a priori" ~ independent of ε

(CD)
$$\begin{cases} u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} \\ u^\varepsilon(x,0) = u_0(x) \end{cases}$$

We have proved the following:

> There exists a solution of (CD), and the derivatives u^ε_x and u^ε_{xx} are bounded for $\varepsilon > 0$:
>
> $$|u_x(x,t)| \le C_{\varepsilon,t} \qquad |u_{xx}(x,t)| \le C_{\varepsilon,t}.$$

The constant $C_{\varepsilon,t}$ depends on t and ε.

and grows as $\varepsilon \to 0$ or $t \to \infty$.

Let u and v be solutions of (CD) with initial data u_0 and v_0.

Entropy / energy methods

$u_t + f(u)_x = \varepsilon u_{xx}$ multiply with $\eta'(u)$, $\eta(0)=0$, η convex

$$\eta'(u)u_t + \eta'(u)f'(u)u_x = \varepsilon\eta'(u)u_{xx} = \varepsilon(\eta'(u)u_x)_x - \varepsilon\eta''(u)(u_x)^2$$

Define $q(u) = \int_0^u \eta'(s)f'(s)\,ds$

$$\eta(u)_t + q'(u)u_x = \varepsilon\eta(u)_{xx} - \eta''(u)(u_x)^2$$

$$\Downarrow$$

$$\eta(u)_t + q(u)_x \le \varepsilon\eta(u)_{xx} \leftarrow \eta''(u) \ge 0$$

↑ ENTROPY INEQUALITY

note (or assume) $\lim_{x \to \pm\infty} u(x,t) = 0$

$\lim_{x \to \infty} u_x(x,t) = 0$

Integrate over $\mathbb{R}$

$$\frac{d}{dt}\int_{\mathbb{R}} \eta(u)\,dx + \underbrace{\int_{\mathbb{R}} q_x(u)\,dx}_{=0} \leq \underbrace{\int_{\mathbb{R}} \varepsilon \eta(u)_{xx}\,dx}_{=0}$$

so that

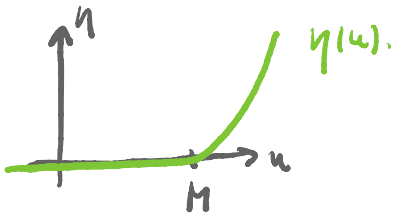
$$\frac{d}{dt}\int_{\mathbb{R}} \eta(u)\,dx \leq 0 \quad \Rightarrow \quad \int_{\mathbb{R}} \eta(u(x,t))\,dx \leq \int_{\mathbb{R}} \eta(u_0(x))\,dx$$

$\eta(u) = |u|^p$, $p \geq 1$ gives

$$\|u(\cdot,t)\|_{L^p(\mathbb{R})} \leq \|u_0\|_{L^p(\mathbb{R})}$$

Let $M \geq \sup_x |u_0(x)|$. Choose η to be any convex function such that

$$\eta(u) = \begin{cases} 0 & u \leq M \\ >0 & u > M \end{cases}$$



We get $\int_{\mathbb{R}} \eta(u(x,t))\,dx = 0$

$\Rightarrow$ $u(x,t) \leq M$ for all x, and all $t > 0$.

We can let $m \leq u_0(x)$ to get

$$m \leq u(x,t).$$

This implies:

$$\operatorname{ess\,inf}_x u_0(x) \leq u(x,t) \leq \operatorname{ess\,sup} u_0(x)$$

MAXIMUM PRINCIPLE