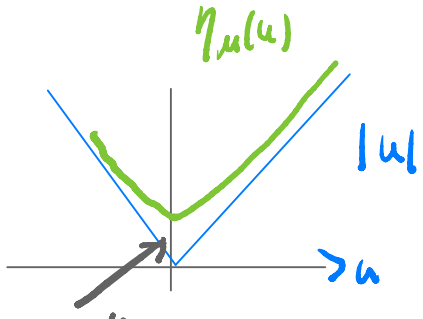
L^1 - stability

Define $\eta_\mu(u) = \sqrt{u^2+\mu^2}$



$\eta_\mu''(u) > 0$

Let u & v be two solutions of (CD) with initial data u_0 & v_0.

$w = u - v$

$$w_t + (f(u) - f(v))_x = \varepsilon (u-v)_{xx}$$

$$A(u,v) = \frac{f(u)-f(v)}{u-v}$$

$$w_t + (Aw)_x = \varepsilon w_{xx}$$

$$\frac{\partial A}{\partial u} = \frac{f'(u)(u-v) - (f(u)-f(v))}{(u-v)^2}$$

$$w_t + Aw_x + A_x w = \varepsilon w_{xx}$$

$= O(1)$, by Taylor series for f

$\eta_\mu'(w)$.

$$\eta_\mu(w)_t + A\eta_\mu(w)_x + A_x \eta_\mu'(w) w = \varepsilon \eta_\mu'(w) w_{xx}$$

$$\eta_\mu(w)_t + (A\eta_\mu(w))_x + A_x(\eta_\mu'(w)w - \eta_A(w)) = \varepsilon \eta_\mu(w)_{xx} - \varepsilon \eta_\mu''(w)(w_x)^2$$

$$\frac{w^2}{\sqrt{w^2+\mu^2}} - \sqrt{w^2+\mu^2} = \frac{-\mu^2}{\sqrt{w^2+\mu^2}}$$

$$\eta_\mu(w)_t + (A\eta(w))_x - \varepsilon \eta_\mu(w)_{xx} \le C_{\varepsilon,t}\,\mu$$

$\eta_\mu(w) \longmapsto |w|$ as $\mu \to 0$

$\mu \to 0$

$$|w|_t + (A|w|)_x \le \varepsilon |w|_{xx}$$

$$A|w| = \operatorname{sgn}(u-v)(f(u)-f(v))$$

KRUŽKOV'S ENTROPY INEQUALITY

$$|u-v|_t + [\operatorname{sgn}(u-v)(f(u)-f(v))]_x \leq \varepsilon |u-v|_{xx}$$

Assume that $u-v \to 0$ and $|u-v|_x \to 0$ as $|x| \to \infty$

Integrate over $x \in \mathbb{R}$ to get

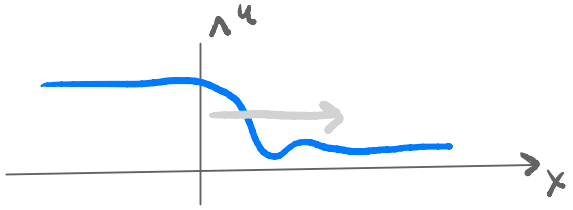
$$\frac{d}{dt} \int_{\mathbb{R}} |u(x,t) - v(x,t)| \, dx \leq 0$$

L^1-contraction:

$$\| u(\cdot,t) - v(\cdot,t) \|_{L^1(\mathbb{R})} \leq \| u_0 - v_0 \|_{L^1(\mathbb{R})}$$

Travelling wave solutions

$$u(x-t) = v(x-st) \qquad s \in \mathbb{R}$$



$$\xi = x - st$$

$$\dot{} = \frac{d}{d\xi}$$

$$-s\dot{v} + \dot{f(v)} = \varepsilon \ddot{v},$$

$$\lim_{\xi \to -\infty} v = u_\ell$$

$$\lim_{\xi \to \infty} v = u_r$$

ODE, boundary conditions:

$$\lim_{|\xi| \to \infty} \dot{v} = 0$$

Integrate from $-\infty$ to ∞

$$s(u_\ell - u_r) = f(u_\ell) - f(u_r)$$ ← Rankine-Hugoniot!

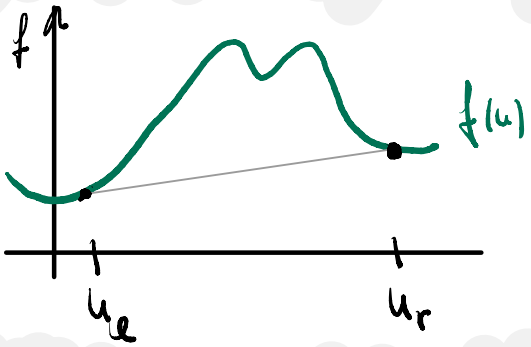
Set $\xi = x/t$ $\quad \frac{d}{d\xi} = 0$ $\quad s = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r}$

$$\dot{v} = f(v) - f(v_\ell) - s(v - u_\ell)$$

ODE, fixpoints u_ℓ, u_r. We want an orbit from u_ℓ to u_r. If $u_\ell < u_r$ then $\dot{v} > 0$ for $\xi \in (u_\ell, u_r)$, i.e.,

$$\frac{f(v) - f(u_\ell)}{v - u_\ell} > \frac{f(u_r) - f(u_\ell)}{u_r - u_\ell} = s \quad , v \in (u_\ell, u_r)$$

slope of secant between (v, u_ℓ) is larger than slope of secant between (u_r, u_ℓ).



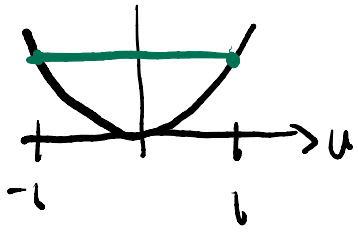
graph of f above line between $(u_\ell, f(u_\ell))$ and $(u_r, f(u_r))$

If $u_\ell > u_r$: graph of $f(u)$ below line between $(u_r, f(u_r))$ and $(u_\ell, f(u_\ell))$.

All this for $u = u^\varepsilon$ solution of (CD). If $u = \lim_{\varepsilon \to 0} u^\varepsilon$, then travelling waves for u will have the same restrictions as those for u^ε!

Example: Travelling wave for $u_t + \frac{1}{2}(u^2)_x = 0$.

$$u_1(x,t) = \begin{cases} -1 & x \le 0 \\ 1 & x \ge 0 \end{cases}$$



u_1 is not a limit of travelling waves for $u^\varepsilon_t + \frac{1}{2}(u^\varepsilon)^2_x = \varepsilon u^\varepsilon_{xx}$!

Convergence of u^ε to u

Theorem A.11 *Let $u_\eta : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be a family of functions such that for each positive T,*

$$(A_1) \qquad |u_\eta(x,t)| \le C_T, \quad (x,t) \in \mathbb{R}^n \times [0,T],$$

for a constant C_T independent of η. Assume in addition that for all compact $B \subset \mathbb{R}^n$ and for $t \in [0,T]$,

$$(A_2) \qquad \sup_{|\xi| \le |\rho|} \int_B |u_\eta(x+\xi,t) - u_\eta(x,t)|\, dx \le \nu_{B,T}(|\rho|),$$

for a modulus of continuity $\nu_{B,T}$. Furthermore, assume that for s and t in $[0,T]$,

$$(A_3) \qquad \int_B |u_\eta(x,t) - u_\eta(x,s)|\, dx \le \omega_{B,T}(|t-s|) \text{ as } \eta \to 0,$$

for some modulus of continuity $\omega_{B,T}$. Then there exists a sequence $\eta_j \to 0$ such that for each $t \in [0,T]$ the sequence $\{u_{\eta_j}(t)\}$ converges to a function $u(t)$ in $L^1_{\text{loc}}(\mathbb{R}^n)$. The convergence is in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}^n))$.

We shall use Thm. A.II to show convergence of the family $\{u^\varepsilon\}_{\varepsilon>0}$ of solutions to

$$(CL_\varepsilon) \qquad \begin{cases} u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} & t>0 \quad x\in\mathbb{R} \\ u^\varepsilon(x,0) = u_0(x) \end{cases}$$

Must show A_1, A_2 and A_3

Maximum principle $\longleftrightarrow$ A.I ✓

A_2: BV seminorm of a function $u = u(x)$

$$|u|_{BV(\mathbb{R})} = \lim_{\rho\to 0}\left(\sup_{|\xi|\le\rho} \int_{\mathbb{R}} |u(x+\xi) - u(x)|\,dx \right)$$

If $|u|_{BV} < \infty$ we say that u has bounded variation

If u has bounded variation, then

$$\int_{\mathbb{R}} |u(x+\rho) - u(x)|\,dx \le |u|_{BV}\,\rho .$$

Assume now that u_0 has bounded variation ($u_0 \in BV(\mathbb{R})$)

Use the L^1- contraction for u^ε, with $v = u^\varepsilon(\cdot+\xi, t)$.

$$\int_{\mathbb{R}} |u^\varepsilon(x+\xi,t) - u^\varepsilon(x,t)|\,dx \le \int_{\mathbb{R}} |u_0(x+\xi) - u_0(x)|\,dx$$

$$\le |\xi|\,|u_0|_{BV(\mathbb{R})} =: \omega(|\xi|)$$

This is A_2.

↑ modulus of continuity.

To prove A_3 we use Kružkov's interpolation lemma

Lemma 4.12 (Kružkov interpolation lemma) *Let $u(x,t)$ be a bounded measurable function defined in the cylinder $\mathcal{B}_{r+\hat{r}} \times [0,T]$, $\hat{r} \geq 0$. For $t \in [0,T]$ and $|\rho| \leq \hat{r}$, assume that u possesses a spatial modulus of continuity*

$$\sup_{|\xi| \leq |\rho|} \int_{\mathcal{B}_r} |u(x+\xi,t) - u(x,t)| \, dx \leq \nu_{r,T,\hat{r}}(|\rho|; u), \tag{4.85}$$

where $\nu_{r,T,\hat{r}}$ does not depend on t. Suppose that for every $\phi \in C_0^\infty(\mathcal{B}_r)$ and $t_1, t_2 \in [0,T]$,

$$\left| \int_{\mathcal{B}_r} (u(x,t_2) - u(x,t_1)) \, \phi(x) \, dx \right| \leq \text{Const}_{r,T} \left(\sum_{|\alpha| \leq m} \|D^\alpha \phi\|_{L^\infty(\mathcal{B}_r)} \right) |t_2 - t_1|, \tag{4.86}$$

where α denotes a multi-index.

Then for t and $t+\tau \in [0,T]$ and for all $\varepsilon \in (0,\hat{r}]$,

$$\int_{\mathcal{B}_r} |u(x,t+\tau) - u(x,t)| \, dx \leq \text{Const}_{r,T} \left(\varepsilon + \nu_{r,T,\hat{r}}(\varepsilon; u) + \frac{|\tau|}{\varepsilon^m} \right). \tag{4.87}$$

u is our ε

We must show (4.85) and (4.86).

4.85 : $\int_{\mathbb{R}} |u^\varepsilon(x+\xi,t) - u^\varepsilon(x,t)| \, dx \leq |u^0|_{BV} \rho$ if $|\xi| \leq \rho$. ✓

4.86

$$\left| \int_{B(0,r)} \left(u^\varepsilon(x,t+\tau) - u^\varepsilon(x,t) \right) \varphi(x) \, dx \right|$$

$$= \left| \int_{B(0,r)} \int_t^{t+\tau} u^\varepsilon_t(x,s) \, ds \, \varphi(x) \, dx \right|$$

$$= \left| \int_{B(0,r)} \int_t^{t+\tau} -f(u^\varepsilon(x,s))_x + \varepsilon u^\varepsilon_{xx}(x,s) \, ds \, \varphi(x) \, dx \right|$$

$$= \left| \int_{B(0,r)} \int_t^{t+\tau} -f(u^\varepsilon(x,s))_x \varphi(x) - \varepsilon u^\varepsilon_x(x,s) \varphi_x(x) \, ds \, dx \right|$$

$$\leq \int_{\mathbb{R}} \int_t^{t+\tau} L |u^\varepsilon_x(x,s)| \varphi(x) + \varepsilon |u^\varepsilon_x(x,s)| \, |\varphi_x(x)| \, ds \, dx$$

$$\leq \int_t^{t+\tau} L |u^\varepsilon(\cdot,s)|_{BV(\mathbb{R})} \|\varphi\|_{L^\infty(\mathbb{R})} + \varepsilon |u^\varepsilon(\cdot,s)|_{BV(\mathbb{R})} \|\varphi_x\|_{L^\infty(\mathbb{R})} \, ds$$

$$\leq C |u_0|_{BV(\mathbb{R})} \left(\|\varphi\|_{L^\infty(\mathbb{R})} + \|\varphi_x\|_{L^\infty(\mathbb{R})} \right) \tau$$

Therefore we can use $\hat{r} = \infty$, $m = 1$ in the interpolation lemma.

$$\int_{\mathbb{R}} |u^\varepsilon(x, t+\tau) - u^\varepsilon(x,t)| \, dx \leq C \left(\mu + \frac{\tau}{\mu} \right) \quad \text{for all } \mu > 0.$$

Choose $\mu = \sqrt{\tau}$ to conclude:

$$\int_{\mathbb{R}} |u^\varepsilon(x, t+\tau) - u^\varepsilon(x,t)| \, dx \leq C \sqrt{\tau}.$$

which is A_3.

Theorem: Assume that $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, and let u^ε solve (CL_ε). Then there exists a function $u \in L^1(\mathbb{R})$ such that $u^\varepsilon \to u$ as $\varepsilon \to 0$. The convergence is in $C([0,\infty); L^1(\mathbb{R}))$.

We must show that the limit u is a **weak solution**, or better, an **entropy solution**.

Definition: A function $u \in L^1(\mathbb{R})$ is called an **entropy solution** of CL_0 if for all non-negative test functions $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$, and for all constants $k \in \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}} |u-k| \varphi_t + q(u,k)\varphi_x \, dx\,dt + \int_{\mathbb{R}} |u_0 - k| \varphi(x,0)\, dx \geq 0$$

$$q(u,k) = \text{sign}(u-k)\,(f(u) - f(k))$$

GO TO NEXT PAGE!

Doubling of variables

Let $u = u(x,t)$ and $v = v(y,s)$ be entropy solutions.

Define $\Psi(x,y,t,s) = \omega_\varepsilon(x-y)\,\omega_\varepsilon(t-s)\,\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$

ω_ε - standard mollifier in 1D

φ - non-negative test function.

Ψ is a non-negative test function in <u>both</u> (x,t) and (y,s)

$$\Psi_t = \omega_\varepsilon(x-y)\,\omega_\varepsilon'(t-s)\,\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) + \omega_\varepsilon(x-y)\,\omega_\varepsilon(t-s)\,\frac{1}{2} D_1\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$$

SKIP NEXT PAGE

Claim: The limit $u = \lim_{\varepsilon \to 0} u^\varepsilon$ is an **entropy solution**.

We have

$$\int_0^\infty \int_{\mathbb{R}} |u^\varepsilon - k| \varphi_t + q(u^\varepsilon, k) \varphi_x \, dx dt + \int_{\mathbb{R}} |u_0^\varepsilon - k| \varphi(x,0) \, dx \geq \varepsilon \int_0^\infty \int_{\mathbb{R}} \varepsilon |u^\varepsilon - k|_x \varphi_x \, dx dt$$

$\downarrow \varepsilon \to 0$ $\qquad$ $\downarrow$?

$$\int_0^\infty \int_{\mathbb{R}} |u - k| \varphi_t + q(u,k) \varphi_x \, dx dt + \int_{\mathbb{R}} |u_0(x) - k| \varphi(x,0) \, dx$$

We want that ? to zero as $\varepsilon \to 0$. $\quad \text{supp}(\varphi) \subset \mathbb{R} \times [0,T]$

$$|?| \leq \varepsilon \int_0^\infty \int_{\mathbb{R}} |u - k|_x |\varphi_x| \, dx dt \leq \varepsilon \|\varphi_x\|_{L^\infty} \int_0^T \int_{\mathbb{R}} |u_x| \, dx dt$$

$$\leq \varepsilon \|\varphi_x\|_{L^\infty} \int_0^T |u(\cdot, t)|_{BV} \, dt \leq \varepsilon \|\varphi_x\|_{L^\infty} T |u_0|_{BV}$$

$$\longrightarrow 0 \text{ as } \varepsilon \to 0.$$

u is an entropy solution

Go to D.o.V.