We must show that the limit $u$ is a **weak solution**, or better, an **entropy solution**.

**Definition:** A function $u \in L^1(\mathbb{R})$ is called an **entropy solution** of $CL_0$ if for all non-negative test functions $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$, and for all constants $k \in \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}} |u-k| \varphi_t + q(u,k)\varphi_x \, dx\,dt + \int_{\mathbb{R}} |u_0 - k| \varphi(x,0)\, dx \geq 0$$

$$q(u,k) = \text{sign}(u-k)\,(f(u) - f(k))$$

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## Doubling of variables

Let $u = u(x,t)$ and $v = v(y,s)$ be entropy solutions.

Define $\Psi(x,y,t,s) = \omega_\varepsilon(x-y)\,\omega_\varepsilon(t-s)\,\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$

$\omega_\varepsilon$ - standard mollifier in 1D

$\varphi$ - non-negative test function.

$\Psi$ is a non-negative test function in <u>both</u> $(x,t)$ and $(y,s)$

$$\Psi_t = \omega_\varepsilon(x-y)\,\omega_\varepsilon'(t-s)\,\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) + \omega_\varepsilon(x-y)\,\omega_\varepsilon(t-s)\,\frac{1}{2} D_1\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$$

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**Claim:** The limit $u = \lim_{\varepsilon \to 0} u^\varepsilon$ is an **entropy solution**.

We have

$$\int_0^\infty \int_{\mathbb{R}} |u^\varepsilon - k| \varphi_t + q(u^\varepsilon, k) \varphi_x \, dx dt + \int_{\mathbb{R}} |u_0^\varepsilon - k| \varphi(x,0) \, dx \geq \varepsilon \int_0^\infty \int_{\mathbb{R}} \varepsilon |u^\varepsilon - k|_x \varphi_x \, dx dt$$

$\varepsilon \to 0$ ↓ ; ? →

$$\int_0^\infty \int_{\mathbb{R}} |u - k| \varphi_t + q(u,k) \varphi_x \, dx dt + \int_{\mathbb{R}} |u_0(x) - k| \varphi(x,0) \, dx$$

We want that ? to zero as $\varepsilon \to 0$. ← $\operatorname{supp}(\varphi) \subset \mathbb{R} \times [0,T]$

$$|?| \leq \varepsilon \int_0^\infty \int_{\mathbb{R}} ||u - k|_x| |\varphi_x| \, dx dt \leq \varepsilon \|\varphi_x\|_{L^\infty} \int_0^T \int_{\mathbb{R}} |u_x| \, dx dt$$

$$\leq \varepsilon \|\varphi_x\|_{L^\infty} \int_0^T |u(\cdot, t)|_{BV} \, dt \leq \varepsilon \|\varphi_x\|_{L^\infty} T |u_0|_{BV}$$

$$\to 0 \text{ as } \varepsilon \to 0.$$

$u$ is an entropy solution

← GO TO D.o.V.

Now we want to show that

$$|u - v|_t + q(u,v)_x \leq 0 \quad \text{weakly}$$

$$\int_0^\infty \int_{\mathbb{R}} |u - v| \varphi_t + q(u,v) \varphi_x \, dx dt + \int_{\mathbb{R}} |u_0 - v_0| \varphi(x,0) \, dx \geq 0$$

for all test functions $\varphi \geq 0$

$$\psi_s = -\omega_\varepsilon(x-y)\,\omega_\varepsilon'(t-s)\,\varphi\left(\tfrac{x+y}{2},\tfrac{t+s}{2}\right) + \omega_\varepsilon(x-y)\,\omega_\varepsilon(t-s)\,\tfrac{1}{2}D_2\varphi\left(\tfrac{x+y}{2},\tfrac{t+s}{2}\right)$$

$$\psi_x = \omega_\varepsilon'(x-y)\,\omega_\varepsilon(t-s)\,\varphi\left(\tfrac{x+y}{2},\tfrac{t+s}{2}\right) + \omega_\varepsilon(x-y)\,\omega_\varepsilon(t-s)\,\tfrac{1}{2}D_1\varphi\left(\tfrac{x+y}{2},\tfrac{t+s}{2}\right)$$

$$\psi_y = -\omega_\varepsilon'(x-y)\,\omega_\varepsilon(t-s)\,\varphi\left(\tfrac{x+y}{2},\tfrac{t+s}{2}\right) + \omega_\varepsilon(x-y)\,\omega_\varepsilon(t-s)\,\tfrac{1}{2}D_1\varphi\left(\tfrac{x+y}{2},\tfrac{t+s}{2}\right)$$

so that:

$$\psi_t + \psi_s = \omega_\varepsilon(x-y)\,\omega_\varepsilon(t-s)\,D_2\varphi\left(\tfrac{x+y}{2},\tfrac{t+s}{2}\right)$$

$$\psi_x + \psi_y = \omega_\varepsilon(x-y)\,\omega_\varepsilon(t-s)\,D_1\varphi\left(\tfrac{x+y}{2},\tfrac{t+s}{2}\right)$$

Since $u$ and $v$ are entropy solutions

(1) $$\int_0^\infty\int_{\mathbb{R}} |u-k_1|\,\psi_t + q(u,k_1)\,\psi_x\,dx\,dt + \int_{\mathbb{R}} |u_0-k_1|\,\psi(x,y,0,s)\,dx \geq 0$$

(2) $$\int_0^\infty\int_{\mathbb{R}} |v-k_2|\,\psi_s + q(v,k_2)\,\psi_y\,dy\,ds + \int_{\mathbb{R}} |v_0-k_2|\,\psi(x,y,t,0)\,dy \geq 0.$$

1) Set $k_1 = v(y,s)$ and $k_2 = u(x,t)$

2) Integrate (1) wrt. $(y,s)$, integrate (2) wrt. $(x,t)$

3) add the results.

to get: ($v = v(y,s)$; $u = u(x,t)$, otherwise $u(x,s)$ etc.)

$$A \quad \int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} |u-v|\, w_\varepsilon(x-y)\, w_\varepsilon(t-s)\, D_2\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right) dx\,dy\,dt\,ds$$

$$B \quad + \int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} q(u,v)\, w_\varepsilon(x-y)\, w_\varepsilon(t-s)\, D_1\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right) dx\,dy\,dt\,ds$$

$$C \quad + \int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} |u_0(x)-v(y,s)|\, w_\varepsilon(-s)\, w_\varepsilon(x-y)\, \varphi\left(\tfrac{x+y}{2}, \tfrac{s}{2}\right) dx\,dy\,ds$$

$$D \quad + \int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} |u(x,t)-v_0(s)|\, w_\varepsilon(t)\, w_\varepsilon(x-y)\, \varphi\left(\tfrac{x+y}{2}, \tfrac{t}{2}\right) dx\,dy\,dt \quad \geq 0$$

Each term is estimated separately.

**A:** We want A to equal

$$\underbrace{\int_0^\infty\int_{\mathbb{R}} |u-v(x,t)|\,\varphi_t(x,t)\, dx\,dt}_{a} = \int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} |u-v(x,t)|\, w_\varepsilon(x-y)\, w_\varepsilon(t-s)\,\varphi_t(x,t)\, dx\,dy\,dt\,ds$$

$A \leq a + |A-a|$, we try to show that $|a-A|$ is small:

$$|a-A| \leq \int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} |v(x,t)-v(y,s)|\, w_\varepsilon(x-y)\, w_\varepsilon(t-s)\left|\varphi_t(x,t) - D_2\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\right| dx\,dt\,dy\,ds$$

$$+ \int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} |v(x,t)-v(y,s)|\, w_\varepsilon(x-y)\, w_\varepsilon(t-s)\,\varphi_t(x,t)\, dx\,dy\,dt\,ds$$

$$=: a_1 + a_2$$

to estimate $a_1$, use that $\varphi \equiv 0$ if $|x|>m$ or $t \geq l$

also $w_\varepsilon(\tau) = 0$ $|\tau|>\varepsilon$, hence the integrand is zero if

$$(x,y,t,s) \notin [-m-\varepsilon, m+\varepsilon]^2 \times [0, l+\varepsilon]^2$$

Set $M = m+\varepsilon$, $T = l+\varepsilon$. We also have that

$$|\varphi_t(x,t) - D_2\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)| \leq \|\varphi_{tx}\|_{L^\infty} \tfrac{1}{2}|x-y| + \|\varphi_{tt}\|_{L^\infty} \tfrac{1}{2}|t-s|$$

which gives:

$$|a_1| \leq 2\|v_0\|_{L^\infty} \int_0^T \int_0^T \int_{-M}^{M} \int_{-M}^{M} w_\varepsilon(x-y)\, w_\varepsilon(t-s)\,(|x-y| + |t-s|)\, dx\,dy\,dt\,ds$$

$$|a_1| \leq C\varepsilon\, 2MT \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

Next

$$a_2 \leq \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} |v(x,t) - v(x,s)|\, w_\varepsilon(x-y)\, w_\varepsilon(t-s)\, \varphi_t(x,t)\, dx\,dy\,dt\,ds$$

$$- \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} |v(x,s) - v(y,s)|\, w_\varepsilon(x-y)\, w_\varepsilon(t-s)\, \varphi_t(x,t)\, dx\,dy\,dt\,ds$$

$$=: a_{21} + a_{22}$$

we estimate $a_{21}$ & $a_{22}$

$\int_{\mathbb{R}} w_\varepsilon(x-y)\,dy = 1$

$$|a_{21}| \leq \|\varphi_t\|_{L^\infty} \int_0^T \int_0^T \int_{\mathbb{R}} |v(x,s) - v(x,t)|\, w_\varepsilon(t-s)\, dx\,dt\,ds$$

$$\leq \|\varphi_t\|_{L^\infty} C_1' \int_0^T \int_0^T |t-s|\, w_\varepsilon(t-s)\, dt\,ds$$

$$|a_{21}| \leq C_1' \varepsilon T \longrightarrow 0 \text{ as } \varepsilon \to 0$$

next $a_{22}$

$$|a_{22}| \le \|\varphi_x\|_{L^\infty} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |v(x,s) - v(x-z,s)|\, \omega_\varepsilon(z)\, dz\, dx\, ds$$

$$\le C \int_0^T \int_{\mathbb{R}} |z|\, \omega_\varepsilon(z)\, |u(\cdot,s)|_{BV(\mathbb{R})}\, dz\, ds$$

$$\le C\, |v_0|_{BV(\mathbb{R})}\, \varepsilon T \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

We have shown:

$$A \le a + O(\varepsilon)$$

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Estimating $B = \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} q(u,v)\, \omega_\varepsilon(x-y)\, \omega_\varepsilon(t-s)\, D^1\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) dx\, dy\, dt\, ds$

we want this to be $b = \int_0^\infty \int_{\mathbb{R}} q(u, v(x,t))\, \varphi_x(x,t)\, dx\, dt$.

$$B \le b + |B - b|$$

shall show that $|B-b| \to 0$ as $\varepsilon \to 0$.

$$B - b = \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} q(u,v)\, \omega_\varepsilon(x-y)\, \omega_\varepsilon(t-s) \left(D^1\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right) - \varphi_x(x,t)\right) dx\, dy\, dt\, ds$$

$$+ \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \left[q(u,v) - q(u, v(x,t))\right] \omega_\varepsilon(x-y)\, \omega_\varepsilon(t-s)\, \varphi_t(x,t)\, dx\, dy\, dt\, ds$$

$$=: b_1 + b_2.$$