next a_{22}

$$|a_{22}| \le \|\varphi_x\|_{L^\infty} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |v(x,s) - v(x-z,s)|\, \omega_\varepsilon(z)\, dz\, dx\, ds$$

$$\le C \int_0^T \int_{\mathbb{R}} |z|\, \omega_\varepsilon(z)\, |u(\cdot,s)|_{BV(\mathbb{R})}\, dz\, ds$$

$$\le C\, |v_0|_{BV(\mathbb{R})}\, \varepsilon T \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

We have shown:

$$A \le a + O(\varepsilon)$$

Estimating $B = \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} q(u,v)\, \omega_\varepsilon(x-y)\, \omega_\varepsilon(t-s)\, D^1\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) dx\, dy\, dt\, ds$

we want this to be $b = \int_0^\infty \int_{\mathbb{R}} q(u, v(x,t))\, \varphi_x(x,t)\, dx\, dt$.

$$B \le b + |B - b|$$

shall show that $|B-b| \to 0$ as $\varepsilon \to 0$.

$$B - b = \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} q(u,v)\, \omega_\varepsilon(x-y)\, \omega_\varepsilon(t-s) \left(D^1\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right) - \varphi_x(x,t)\right) dx\, dy\, dt\, ds$$

$$+ \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \left[q(u,v) - q(u, v(x,t))\right] \omega_\varepsilon(x-y)\, \omega_\varepsilon(t-s)\, \varphi_t(x,t)\, dx\, dy\, dt\, ds$$

$$=: b_1 + b_2.$$

b_1:

Since φ has compact support, we can find M, T such that

$$D^1\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) - \varphi_t(x,t) = 0 \quad \text{if } s>T \text{ or } t>T \text{ or } |y|>M \text{ or } |x|>M.$$

Furthermore:

$$\left|D^1\varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) - \varphi_t(x,t)\right| \le \|\varphi_{tx}\|_{L^\infty}\frac{1}{2}|t-s| + \|\varphi_x\|_{L^\infty}\frac{1}{2}|x-y|$$

Also: since u and v are bounded $|q(u,v)| \le M$

Then we get:

$$|b_1| \le M\int_0^T\int_0^T\int_{-M}^{M}\int_{-M}^{M} \omega_\varepsilon(x-y)\,\omega_\varepsilon(t-s)\,C(|x-y|+|t-s|)\,dx\,dy\,dt\,ds$$

$$\le M\varepsilon C\int_0^T\int_{-M}^{M} dx\,dt = C\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

b_2

$$b_2 = \int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} \left[q(u, v(y,s)) - q(u, v(x,s))\right]\omega_\varepsilon(t-s)\,\omega_\varepsilon(x-y)\,\varphi_x(x,t)\,dx\,dy\,dt\,ds$$

$$+ \int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} \left[q(u, v(x,s)) - q(u, v(x,t))\right]\omega_\varepsilon(x-y)\,\omega_\varepsilon(t-s)\,\varphi_{\not t}(x,t)\,dx\,dy\,dt\,ds$$

$$=: b_{21} + b_{22}.$$

Observe that

$$|q(a,b) - q(a,c)| \le L|a-c|, \quad L = |f|_{Lip}$$

b_{21}

$$|b_{21}| \le L \|\varphi_x\|_{L^\infty} \int_0^T \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |v(x,s) - v(y,s)| \, w_\varepsilon(x-y) \, w_\varepsilon(t-s) \, dx\,dy\,dt\,ds$$

$z = x - y$

$$= L \|\varphi_x\|_{L^\infty} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |v(y+z,s) - v(y,s)| \, w_\varepsilon(z) \, dz\,dy\,ds$$

$$\le L \|\varphi_x\|_{L^\infty} \int_0^T \int_{\mathbb{R}} |z| \, |v(\cdot,s)|_{BV(\mathbb{R})} \, w_\varepsilon(z) \, dz\,ds$$

$$\le L \|\varphi_x\|_{L^\infty} |v_0|_{BV(\mathbb{R})} \int_0^T \int_{\mathbb{R}} |z| \, w_\varepsilon(z) \, dz\,ds$$

$$\le L \|\varphi_x\|_{L^\infty} |v_0|_{BV(\mathbb{R})} T \varepsilon \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

b_{22}

$$|b_{22}| \le \|\varphi_x\|_{L^\infty} \int_0^T \int_0^T \int_{-M}^{M} \int_{-M}^{M} |v(x,s) - v(x,t)| \, w_\varepsilon(x-y) \, w_\varepsilon(t-s) \, dx\,dy\,dt\,ds$$

$$\le \|\varphi_x\|_{L^\infty} \int_0^T \int_0^T \int_{-M}^{M} |v(x,t) - v(x,s)| \, dx \, w_\varepsilon(t-s) \, dt\,ds$$

$$\le C' \int_0^T \int_0^T |t-s| \, w_\varepsilon(t-s) \, dt\,ds$$

$$\le C \varepsilon \int_0^T \int_0^T w_\varepsilon(t-s) \, dt\,ds \le C' T \varepsilon \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

C and D are estimated in the same way, we do C.

We want C to be

$$\frac{1}{2} \int_{\mathbb{R}} |u_0(x) - v_0(x)| \, \varphi(x,0) \, dx =: C$$

$$C \le C + |C - c|$$

$$|C - c| = \left| \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} |u_0(x) - v(y,s)| \, \omega_\varepsilon(s) \omega_\varepsilon(x-y) \left(\varphi\left(\tfrac{x+y}{2}, \tfrac{s}{2}\right) - \varphi(x,0) \right) dx\,dy\,ds \right|$$

$$+ \left| \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \left(|u_0(x) - v(y,s)| - |u_0(x) - v_0(x)| \right) \omega_\varepsilon(x-y) \, \omega_\varepsilon(s) \, \varphi(x,0) \, dx\,dy\,ds \right|$$

$$=: C_1 + C_2$$

C_1

$$|C_1| \le K \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \omega_\varepsilon(s) \omega_\varepsilon(x-y) \left(\tfrac{s}{2} + \tfrac{|x-y|}{2} \right) dx\,dy\,ds$$

$$\le K \, \varepsilon/2 \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

C_2

$$|C_2| \le \|\varphi\|_{L^\infty} \int_0^\varepsilon \int_{\mathbb{R}} \int_{\mathbb{R}} |v(y,s) - v_0(x)| \, \omega_\varepsilon(x-y) \, \omega_\varepsilon(s) \, dx\,dy\,ds$$

$$\le \|\varphi\|_{L^\infty} \int_0^\varepsilon \int_{\mathbb{R}} \int_{\mathbb{R}} |v(y,s) - v_0(y)| \, \omega_\varepsilon(x-y) \, \omega_\varepsilon(s) \, dx\,dy\,ds$$

$$+ \|\varphi\|_{L^\infty} \int_0^\varepsilon \int_{\mathbb{R}} \int_{\mathbb{R}} |v_0(x) - v_0(y)| \, \omega_\varepsilon(x-y) \, \omega_\varepsilon(s) \, dx\,dy\,ds$$

$$=: C_{21} + C_{22}$$

C_{21}

$$|C_{21}| \le K \int_0^\varepsilon s\, \omega_\varepsilon(s)\, ds = K\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

C_{22}

$$|C_{22}| \le K \int_0^\varepsilon \int_{\mathbb{R}} \int_{\mathbb{R}} |v_0(y+z) - v_0(y)|\, dy\, \omega_\varepsilon(z)\, dz\, \omega_\varepsilon(s)\, ds$$

$$\le K\, |v_0|_{BV(\mathbb{R})} \int_{\mathbb{R}} |z|\, \omega_\varepsilon(z)\, dz \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

Now we have shown:

If u and v are entropy solutions, initial values u_0 and v_0 resp., and

$$|u(\cdot,t)|_{BV(\mathbb{R})} \le |u_0|_{BV(\mathbb{R})}, \quad |v(\cdot,t)|_{BV(\mathbb{R})} \le |v_0|_{BV(\mathbb{R})}$$

$u(\cdot,t), v(\cdot,t) \in L^1(\mathbb{R})$.

Then:

$$|u-v|_t + q(u,v)_x \le 0 \quad \text{weakly.}$$

For any two entropy solutions u and v, we have

$$\int_0^\infty \int_{\mathbb{R}} |u-v|\varphi_t + q(u,v)\varphi_x \, dx\, dt + \int_{\mathbb{R}} |u_0 - v_0|\, \varphi(x,0)\, dx \ge 0$$

$q(u,v) = \operatorname{sgn}(u-v)(f(u) - f(v))$

for all test functions $\varphi \ge 0$

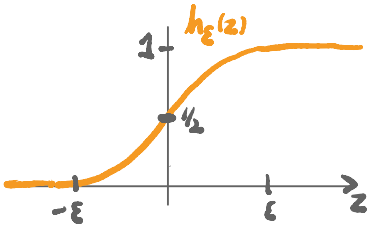
Choosing $\varphi(x,t)$

Let L be the Lipschitz constant of f, we also have that

$$|q(u,v)| \le L|u-v|.$$

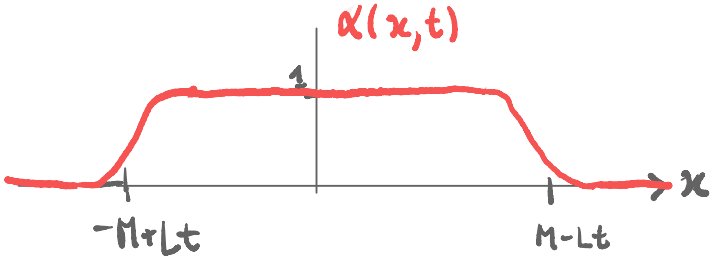
Pick an $M>0$ and $T>0$ such that $LT<M$.

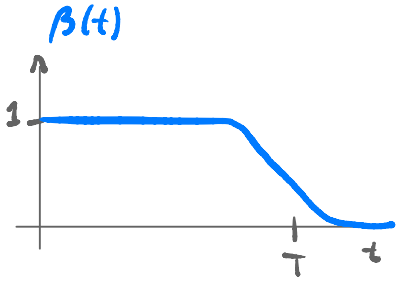
Define: $h_\varepsilon(z) = \int_{-\infty}^{z} \omega_\varepsilon(s)\,ds$



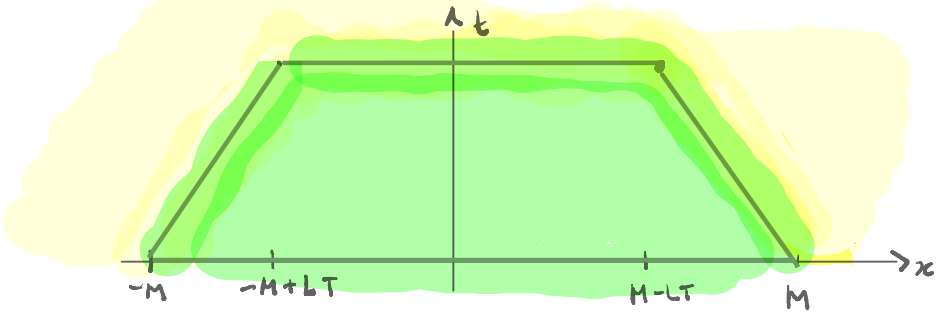
$$\alpha(x,t) = h_\varepsilon(x-Lt+M) - h_\varepsilon(x+Lt-M)$$

$$\beta(t) = h_\varepsilon(T-t)$$





Define: $\varphi(x,t) = \alpha(x,t)\beta(t)$



$\varphi = 0$

$\varphi = 1$

φ is an admissible test function.

$$\varphi_t(x,t) = -L\left(\omega_\varepsilon(x-Lt+M) + \omega_\varepsilon(x+Lt-M)\right)\beta(t)$$

$$- \alpha(x,t)\,\omega_\varepsilon(T-t)$$

$$\varphi_x(x,t) = \left(\omega_\varepsilon(x-Lt+M) + \omega_\varepsilon(x-Lt-M)\right)\beta(t)$$

$$|u-v|\varphi_t + q(u,v)\varphi_x = -\alpha(x,t)\,\omega_\varepsilon(T-t)\,|u-v|$$

$$-L\left(\omega_\varepsilon(x-Lt+M)+\omega_\varepsilon(x+Lt-M)\right)\beta(t)\,|u-v|$$

$$+\left(\omega_\varepsilon(x-Lt+M)+\omega_\varepsilon(x+Lt-M)\right)\beta(t)\,q(u,v)$$

$|q(u,v)| \le L|u-v|$

$$\le -\alpha(x,t)\,\omega_\varepsilon(T-t)\,|u-v|$$

$$+\left(\omega_\varepsilon(x-Lt+M)+\omega_\varepsilon(x+Lt-M)\right)\beta(t)\,|u-v|\underbrace{(-L+L)}_{=0}$$

Therefore

$$\int_0^\infty\int_{\mathbb{R}} |u(x,t)-v(x,t)|\,\omega_\varepsilon(T-t)\,\alpha(x,t)\,dx\,dt$$

$$\le \int_{\mathbb{R}} |u_0-v_0|\,\beta(0)\,(h_\varepsilon(x-M)+h_\varepsilon(x+M))\,dx$$

$\beta(0)=1$

Write this as $L \le R$, and set

$$l = \int_{\mathbb{R}} |u(x,T)-v(x,T)|\,\chi(x)\,dx, \qquad r = \int_{-M}^{M} |u_0(x)-v_0(x)|\,dx$$

where $\chi(x)$ is the characteristic function of the interval

$[-M+LT, M-LT]$

$$|L-l| \le \int_{\mathbb{R}}\int_0^\infty \left(|u(x,T)-v(x,T)|\,\chi(x) - |u(x,t)-v(x,t)|\,\alpha(x,t)\right)\beta(t)\,dx\,dt$$

$$\le \int_0^\infty\int_{\mathbb{R}} \left(|u(x,T)-v(x,T)| - |u(x,t)-v(x,t)|\right)\chi(x)\,\beta(t)\,dx\,dt$$

$$+\int_0^\infty\int_{\mathbb{R}} |u(x,t)-v(x,t)|\,|\chi(x)-\alpha(x,t)|\,\beta(t)\,dx\,dt$$

$$=: a + b$$

Exercises:

1) Show that $a \leq C_1 \varepsilon$

2) Show that $b \leq C_2 \varepsilon$

3) Show that $|R - r| \leq C_3 \varepsilon$.

∴ By letting $\varepsilon \to 0$, we have shown that $\ell \leq r$, i.e.,

$$\int_{-M+LT}^{M-LT} |u(x,T) - v(x,T)|\,dx \leq \int_{-M}^{M} |u_0(x) - v_0(x)|\,dx$$

for any two entropy solutions u and v

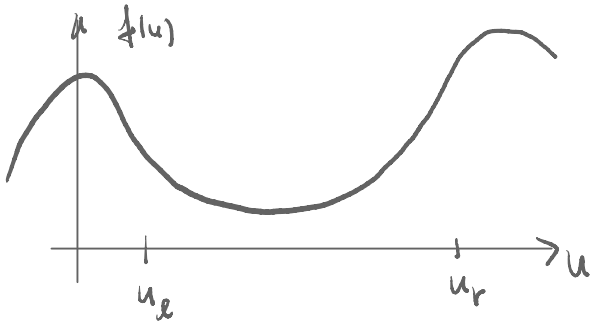
We can, if the limits exist, let $M \to \infty$ to get

$$\|u(\cdot,T) - v(\cdot,T)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})}.$$

The Riemann problem

$$u_t + f(u)_x = 0 \qquad t > 0$$

$$u(x,0) = \begin{cases} u_\ell & x < 0 \\ u_r & x \geq 0 \end{cases}$$



Look for solutions on the form: $u(x,t) = U\left(\frac{x}{t}\right)$

$\xi = \frac{x}{t}$

$$-\frac{x}{t^2}U' + \frac{1}{t}\frac{d}{d\xi}f(U) = 0$$

$$-\xi U'(\xi) + f'(u(\xi))U'(\xi) = 0$$

$$U'(\xi)\left(f'(u) - \xi\right) = 0$$

If $U' \neq 0$ (u not constant), then $\boxed{f'(u(\xi)) = \xi}$

Must invert this formula, i.e. solve for u.

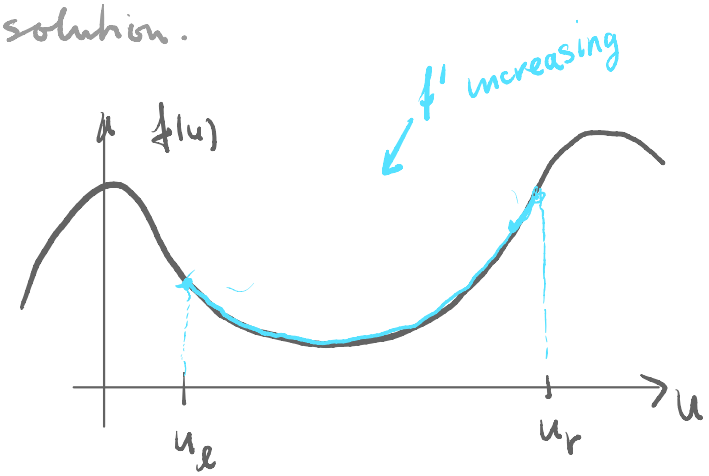
Example: $f(u) = \frac{1}{2}u^2$ $f'(u) = u$, $\rightarrow u(\xi) = \xi$, $u(x,t) = \frac{x}{t}$

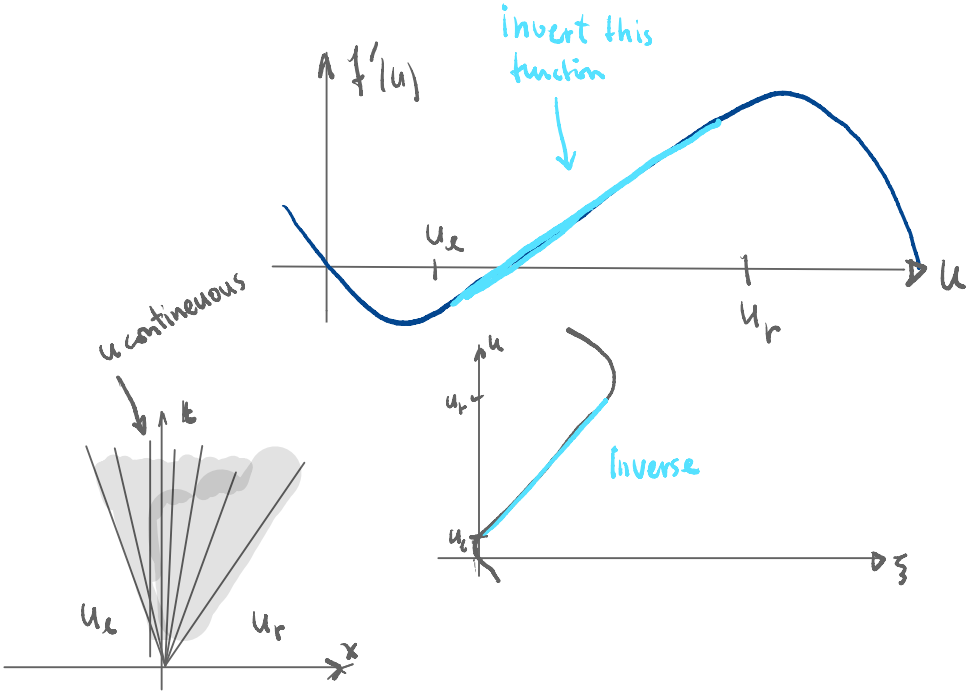
Does not fit initial data!

$u_l < u_r$ If f' is increasing on (u_l, u_r), then we can invert on this interval to get:

$$u(x,t) = \begin{cases} u_l & x < f'(u_l)t \\ (f')^{-1}\left(\frac{x}{t}\right) & tf'(u_l) \leq x \leq tf'(u_r) \\ u_r & tf'(u_r) < x \end{cases}$$

$u_r < u_l$ If f' is decreasing in the interval (u_r, u_l), we can invert to get a solution.





This is called a **Rarefaction wave**

The general solution:

Define $f_\smile(u)$ to be the lower convex envelope of f

$$f_\smile(u) = \sup\{g(u) \mid g \text{ convex on } [u_\ell, u_r],\ g(u) \le f(u)\}$$

$f_\smile(u)$ depends on u_ℓ, u_r!

$f_\frown(u)$ is the upper concave envelope of f

$$f_\frown(u) = \inf\{g \mid g \text{ is concave on } (u_r, u_\ell) \text{ and } g(u) \ge f(u)\}$$

