CLAIM

ENTROPY SOLUTION

If $u_l < u_r$ the solution of the Riemann problem is:

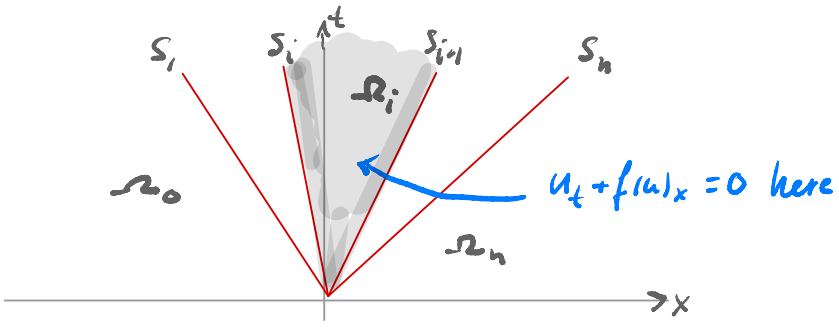
$$u(x,t) = \begin{cases} u_l & x \le f'(u_r)t \\ (f')^{-1}\left(\frac{x}{t}\right) & f'(u_l)t \le x \le f'(u_r)t \\ u_r & f'(u_r)t < x \end{cases}$$

If $u_r < u_l$, replace f with $f^\wedge$

Proof of claim

We show that u is an entropy solution.

Note that u is piecewise differentiable, with jump discontinuities at $x = s_i t$ $i = 1, \dots, n$.



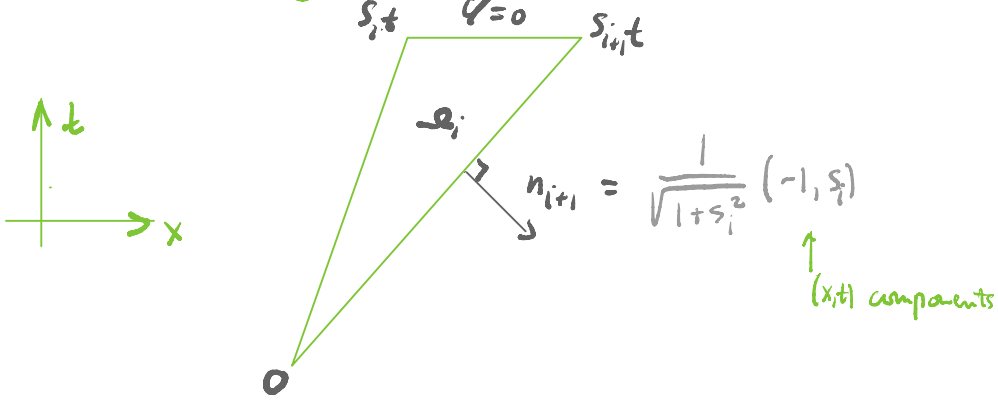
Let $\varphi \ge 0$ be a test function. $\operatorname{Supp}\varphi \subset (0,T)$.
η is a convex continuous entropy. Then

$$\int_0^T \int_{\mathbb{R}} \eta(u)\varphi_t + q(u)\varphi_x \, dx dt = \sum_{i=0}^{n} \iint_{\Omega_i} \eta(u)\varphi_t + q(u)\varphi_x \, dx dt \quad \text{IBP}$$

$$= -\sum_{i=0}^{n} \iint_{\Omega_i} \varphi(\eta_t + q_x)\, dx dt + \sum_{i=0}^{n} \int_{\partial\Omega_i} (\eta, q)\cdot n_i \, dS$$

"0"

n_i - normal.



$$= \sum_{i=1}^{n+1} \int_0^T \left\{ s_i [\![\eta(x_i,t)]\!]_i - [\![q(x_i,t)]\!]_i \right\} \varphi(x_i,t)\, dt \, \frac{1}{\sqrt{1+s_i^2}}$$

$$[\![a]\!]_i := a_i - a_{i-1} \qquad x = x_i(t) = s_i t$$

We shall show that $s[\eta]_i - [\![q]\!]_i \geq 0$ for all i.

Write $u_\ell = u_{i-1}$, $u_r = u_i$, $s = s_i$ etc. Assume that $u_\ell < u_r$.

By construction of u

$$\frac{f(u) - f(u_\ell)}{u - u_\ell} > s > \frac{f(u_r) - f(u)}{u_r - u} \qquad \forall u \in (u_\ell, u_r)$$

Taylor's formula with remainder term for η

$$\eta_r - \eta_\ell = \eta_\ell'(u_r - u_\ell) + \int_{u_\ell}^{u_r} \eta''(u)(u_r - u)\, du \qquad \eta'' \geq 0$$

so that

$$\eta_\ell'(u_r - u_\ell) = \eta_r - \eta_\ell - \int_{u_\ell}^{u_r} \eta''(u)(u_r - u).$$

From Rankine-Hugoniot condition

$$s(u_r - u_\ell) - (f_r - f_\ell) = 0$$

| multiply with η_ℓ'

$$0 = s\eta_\ell'(u_r - u_\ell) - \eta_\ell'(f_r - f_\ell)$$

$$= s(\eta_r - \eta_\ell) - \eta_\ell'(f_r - f_\ell) - \int_{u_\ell}^{u_r} s\eta''(u)(u_r - u)\, dy$$

$$= \underbrace{s(\eta_r - \eta_\ell) - (q_r - q_\ell)}_{\text{This we want to show is negative}} + \underbrace{(q_r - q_\ell) - \eta_\ell'(f_r - f_\ell) - \int_{u_\ell}^{u_r} s\eta''(u)(u_r - u)\, du}_{\text{This we want to be negative} \; =: E}$$

If $E \leq 0$, then

$s[\eta] - [\![q]\!] = -E \geq 0$.

Recall that $q' = \eta' f'$.

$$E = \int_{u_\ell}^{u_r} \eta'(u) f'(u)\,du - \int_{u_\ell}^{u_r} \eta'(u_\ell) f'(u)\,du - \int_{u_\ell}^{u_r} s\,\eta''(u)(u_r - u)\,du$$

$$-s \le -\frac{f(u_r) - f(u)}{u_r - u}$$

$$\le \int_{u_\ell}^{u_r} (\eta'(u) - \eta'(u_\ell)) f'(u)\,du - \int_{u_\ell}^{u_r} \eta''(u)(f(u_r) - f(u))\,du$$

$$= \int_{u_\ell}^{u_r} \int_{u_\ell}^{u} \eta''(v) f'(u)\,dv\,du - \int_{u_\ell}^{u_r} \int_{u}^{u_r} \eta''(u) f'(v)\,dv\,du = 0 \quad !$$

Therefore $s_i [\![\eta_i]\!] - [\![q_i]\!] \ge 0$ and

$$\int_0^\infty \int_{\mathbb{R}} \eta(u)\varphi_t + q(u)\varphi_x\,dx\,dt \ge 0$$

Prove the case where $u_r < u_\ell$

and u is an entropy solution of the Riemann problem. ∎

Exercise (example):

Solve the problem

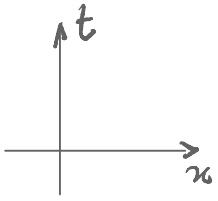
$$\begin{cases} u_t + \frac{1}{2}(u^2)_x = 0, \\ u(x,0) = \begin{cases} 0 & x<0, \\ 1 & 0 \le x \le 1, \\ 0 & x>1. \end{cases} \end{cases}$$

Hint: Start by solving the 2 Riemann problems defined by the initial data. Then solve the i.v.p. for $t > 2$.

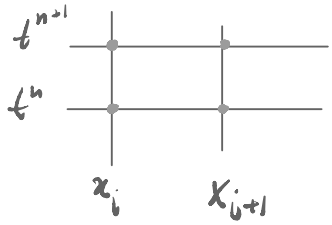
Numerical methods

Assume $f'(u) \ge 0$, then if u is the solution of the Riemann problem defined by (u_ℓ, u_r),

$u(0,t) = u_L$ Why?



Grid



$x_i = i\Delta x$

$t^n = n\Delta t$

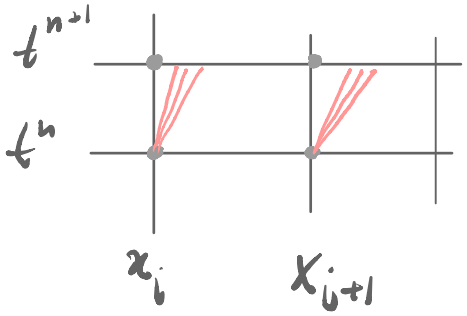
$$u_i^n \leftarrow \text{(approximation to)} \ \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} u(x,t^n)\,dx$$

Let u solve

$$u_t + f(u)_x = 0 \qquad t \in (0, \Delta t)$$

$$u(x,0) = \sum_i u_i^n \chi_{[x_i, x_{i+1}]}(x)$$

Choose Δt so small that waves do not interact for $t < \Delta t$. $L\Delta t \le \Delta x$ Why is this possible?



Set $$u_i^{n+1} = \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} u(x,\Delta t)\,dx$$

$$= \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} u(x,0)\,dx + \frac{1}{\Delta x}\int_0^{\Delta t} f(u(x_i,t))\,dt - \frac{1}{\Delta x}\int_0^{\Delta t} f(u(x_{i+1},t))\,dt$$

$$= u_i^n - \frac{1}{\Delta x}\int_0^{\Delta t} f(u(x_{i+1}-,t))\,dt + \frac{1}{\Delta x}\int_0^{\Delta t} f(u(x_i+,0))\,dt$$

The upwind method.

$$\begin{cases} u_i^{n+1} = u_i^n - \dfrac{\Delta t}{\Delta x}\left[f(u_i^n) - f(u_{i-1}^n) \right] \\ u_i^0 = \dfrac{1}{\Delta x}\displaystyle\int_{x_i}^{x_{i+1}} u_0(x)\,dx \end{cases}$$

Can write this as $\dfrac{u_i^{n+1} - u_i^n}{\Delta t} + \dfrac{f_i^n - f_{i-1}^n}{\Delta x} = 0$

or $u_i^{n+1} = u_i^n - \lambda (f_i^n - f_{i-1}^n) \qquad \lambda = \dfrac{\Delta t}{\Delta x}.$

Set $a_{i-1/2}^n = \lambda \dfrac{f_i^n - f_{i-1}^n}{u_i - u_{i-1}} \geq 0 \quad , \quad a_{i-1/2}^n \leq 1$ CFL

so that the scheme can be written

$$u_i^{n+1} = (1 - a_{i-1/2}^n) u_i^n + a_{i-1/2}^n u_{i-1}^n$$

This is a convex combination (= average) of u_i^n and u_{i-1}^n

We then have

$$\min_i u_i^n \leq \min\{u_i^n, u_{i-1}^n\} \leq u_i^{n+1} \qquad \text{and}$$

$$u_i^{n+1} \leq \max\{u_i^n, u_{i-1}^n\} \leq \max_i u_i^n$$

This implies the discrete maximum principle

$$\inf_x u_x^0 \leq \inf_i u_i^0 \leq u_i^n \leq \sup_i u_i^0 \leq \sup_x u_0(x)$$

We write $u^n = \{u_i^n\} = \{\dots, u_0, u_1, \dots\}$, and define $G(u^n)$ by

$$G(u^n)_i = u_i^n - \lambda(f(u_i^n) - f(u_{i-1}^n))$$

G is monotone

$$\frac{\partial G}{\partial u_i^n} = 1 - \lambda f'(u_i^n) \geq 0 \qquad \frac{\partial G}{\partial u_{i-1}^n} = \lambda f'(u_{i-1}^n) \geq 0$$

G is conservative $\sum_i G(u^n)_i = \sum_i u_i^n$