Crandall-Tartar lemma

$D \subset \Omega$ ← measure space such that $\varphi, \psi \in D \Rightarrow \varphi \vee \psi$ and $\varphi \wedge \psi \in D$

Assume that we have a map $T: D \to D$ such that

$$\int T(\varphi) = \int \varphi .$$

$a \vee b = \max(a,b)$
$a \wedge b = \min(a,b)$

Then the following are equivalent:

1) $\varphi \leq \psi \Rightarrow T(\varphi) \leq T(\psi)$

2) $\int_\Omega (T(\varphi) - T(\psi))^+ \leq \int_\Omega (\varphi - \psi)^+$

3) $\int_\Omega |T(\varphi) - T(\psi)| \leq \int_\Omega |\varphi - \psi|$

Proof

Assume 1). $T(\varphi \vee \psi) - T(\varphi) \geq 0$, $T(\varphi) - T(\psi) \leq T(\psi \vee \varphi) - T(\varphi)$

$\Rightarrow (T(\psi) - T(\varphi))^+ \leq T(\varphi \vee \psi) - T(\psi)$, so that

$$\int_\Omega (T(\varphi) - T(\psi))^+ \leq \int T(\varphi \vee \psi) - T(\psi) = \int (\psi \vee \varphi) - \int \psi = \int (\varphi - \psi)^+$$

Assume 2)

$$\int_\Omega |T(\varphi) - T(\psi)| = \int_\Omega (T(\varphi) - T(\psi))^+ + (T(\psi) - T(\varphi))^+$$
$$\leq \int_\Omega (\varphi - \psi)^+ + (\psi - \varphi)^+ = \int_\Omega |\varphi - \psi|$$

Assume 3) and that $\varphi < \psi$.

$$\int_\Omega (T(\varphi) - T(\psi))^+ = \int_\Omega \tfrac{1}{2} |T(\varphi) - T(\psi)| + \int_\Omega \tfrac{1}{2} (T(\varphi) - T(\psi))$$
$$\leq \tfrac{1}{2} \int_\Omega |\varphi - \psi| + (\varphi - \psi) = 0 \qquad \blacksquare$$

Since G is conservative we can use the C-T lemma

with $\psi = u^n$ $\psi = v^n$ ← another solution, with initial data v_i^0

to deduce that

$$\sum_i |u_i^{n+1} - v_i^{n+1}| \leq \sum_i |u_i^n - v_i^n|$$

and by induction

$$\sum_i |u_i^n - v_i^n| \leq \sum_i |u_i^0 - v_i^0| \leq \frac{1}{\Delta x} \int_{\mathbb{R}} |u_0(x) - v_0(x)| dx$$

Define the piecewise constant function

$$u_{\Delta t}(x,t) = \sum_{i,n} u_i^n \chi_{[x_i, x_{i+1})}(x) \chi_{[t^n, t^{n+1})}(t)$$

Use $v_i^n = u_{i-1}^n$

$$|u_{\Delta t}(\cdot, t^{n+1})|_{BV} = \sum_i |u_i^{n+1} - u_{i-1}^{n+1}| \leq \sum_i |u_i^0 - u_{i-1}^0|$$
$$= |u_{\Delta t}(\cdot, 0)|_{BV}$$

Use $v_i^n = u_i^{n+1}$

$$\sum_i |u_i^{n+1} - u_i^n| \leq \sum_i |u_i^1 - u_i^0| = \lambda \sum_i |f(u_i^0) - f(u_{i-1}^0)|$$

Multiply with Δx

$$\|u_{\Delta t}(\cdot, t^{n+1}) - u_{\Delta t}(\cdot, t^n)\|_{L^1(\mathbb{R})} = \Delta x \sum_i |u_i^{n+1} - u_i^n|$$

$$\leq \Delta t \, |f|_{Lip} \sum_i |u_i^0 - u_{i-1}^0| = \Delta t \, |f|_{Lip} \, |u_{\Delta t}(\cdot, 0)|_{BV}$$

which implies

$$\| u_{\Delta t}(\cdot, t^n) - u_{\Delta x}(\cdot, t^m) \|_{L^1(\mathbb{R})}$$

$$\leq (n-m)\Delta t \, |f|_{Lip} \, |u_0|_{BV}$$

We can now use Theorem A11 and conclude that there is a function $u \in C([0,\infty); L^1(\mathbb{R}))$ such that

$$u_{\Delta t_j} \longrightarrow u \quad \text{in} \quad C([0,\infty); L^1(\mathbb{R}))$$

and $|u(\cdot,t)|_{BV} \leq |u_0|_{BV}$.

Exercise Show that u is an entropy solution.

Let G be as above.

a) Show that $G(u \vee c) \leq G(u) \vee c$ and that $G(u \wedge c) \geq G(u) \wedge c$

b) Show that

$$|u_i^{n+1} - c| \leq |u_i^n - c| + \lambda \left(q(u_i^n, c) - q(u_{i-1}^n, c) \right) \leq 0$$

where $q(u,c)$ is the Kruzkov entropy flux

$$q(u,c) = \operatorname{sgn}(u-c)\left(f(u) - f(c)\right) = f(u \vee c) - f(u \wedge c).$$

c) Show that $u = \lim_{\Delta t \to 0} u_{\Delta t}$ is an entropy solution.

Please do this, you can ask me...