Conservation laws

$u = u(x,t)$ for simplicity we work with $x \in \mathbb{R}$



$$u_t + f(u)_x = 0$$

$f \in C^2(\mathbb{R})$, the flux function

$$m(t) = \int_a^b u(x,t)\,dx$$ ← Amount of "stuff" between a and b.

$$m'(t) = \int_a^b u_t(x,t)\,dx$$

$$= \int_a^b -f(u(x,t))_x\,dx$$

$$= -f(u(b,t)) + f(u(a,t))$$

$f(u(b,t))$: inflow at b

$f(u(a,t))$: outflow at a

Conservation law:

Change of mass in a interval = inflow − outflow.

very general model:

- mass, momentum, energy, ...

Fluid dynamics

The Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0 & t>0,\ x\in\mathbb{R} \\ u(x,0) = u_0(x) \end{cases}$$

The linear case: $f(u) = au$

$$u_t + a u_x = 0$$

Characteristics: $X'(t) = a$, $X(0) = x_0$

$$\Rightarrow X(t) = at + x_0$$

$$\frac{d}{dt} u(X(t),t) = u_t + u_x X'(t) = u_t(X(t),t) + a u_x(X(t),t) = 0.$$

$$u(X(t),t) = u_0(x_0)$$

or

$$\boxed{u(x,t) = u_0(x-at)}$$

In general: $f(u)_x = f'(u) u_x$

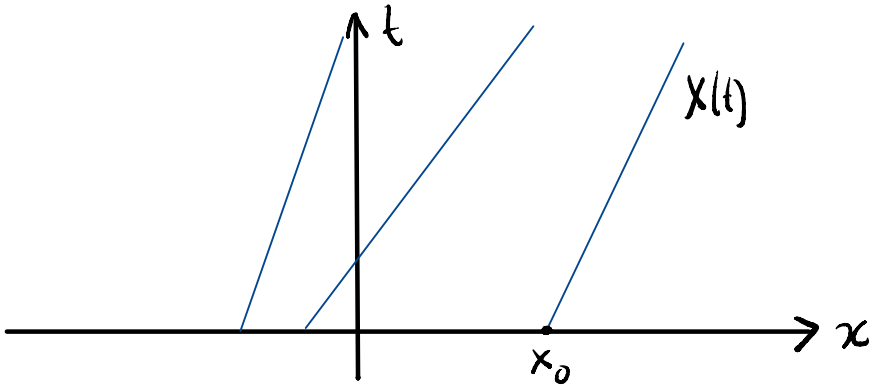
$$u_t + f'(u) u_x = 0, \quad \text{characteristics } X'(t) = f'(u(X(t),t))$$

$$\frac{d}{dt} u(X(t),t) = 0 \Rightarrow u(X(t),t) \text{ is constant!}$$

$f'(u(X(t),t))$ is constant

$t \mapsto X(t)$ is a straight line

$$X(t) = x_0 + f'(u_0(x_0))t$$



Assume that we can solve

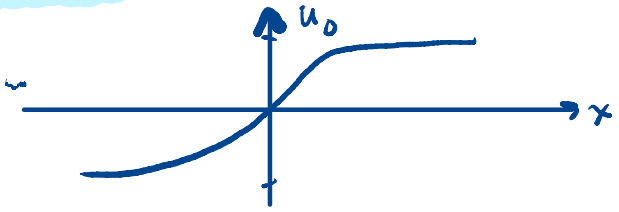
$x = f'(u_0(x_0))\,t + x_0$ to get $x_0 = \xi(x,t)$

then $u(x,t) = u_0(\xi(x,t))$

we can solve if $x_0 \longmapsto t f'(u_0(x_0)) + x_0$ is monotone

Example

$f(u) = \frac{1}{2}u^2$ $\quad f'(u) = u$ $\quad u_0(x) = \tanh^{-1}(x)$

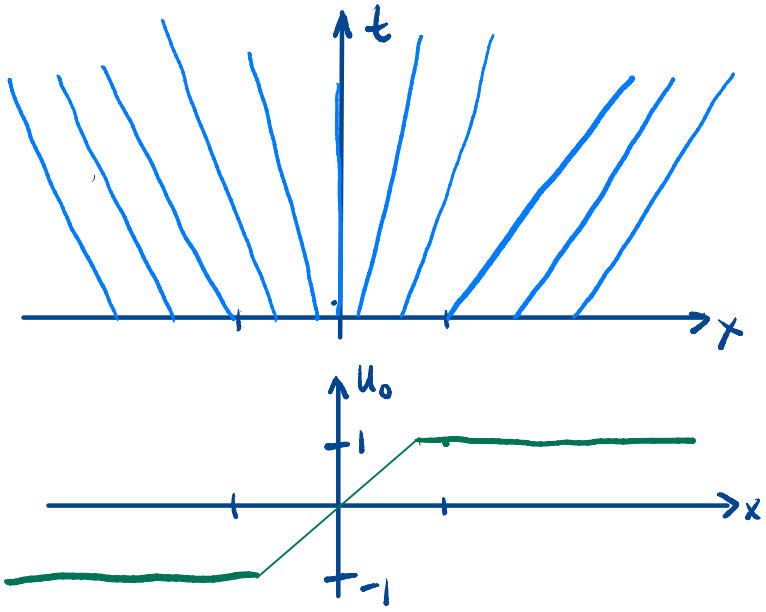


O.K.

$$\frac{d}{dy}\left(y + t\,u_0(y)\right) = 1 + \frac{1}{\cosh^2(y)} > 0$$

Difficult to get explicit formula...

$$u_0(x) = \begin{cases} -1 & x \le -1 \\ x & -1 < x < 1 \\ 1 & x > 1 \end{cases} \Rightarrow X(t) = \begin{cases} x_0 - t & x_0 \le -1 \\ x_0 + x_0 t & x_0 \in (-1,1) \\ x_0 + t & x_0 \ge 1 \end{cases}$$



We get a formula for $u(x,t)$

$$u(x,t) = u_0\left(\frac{x}{t}\right)$$

Checking...

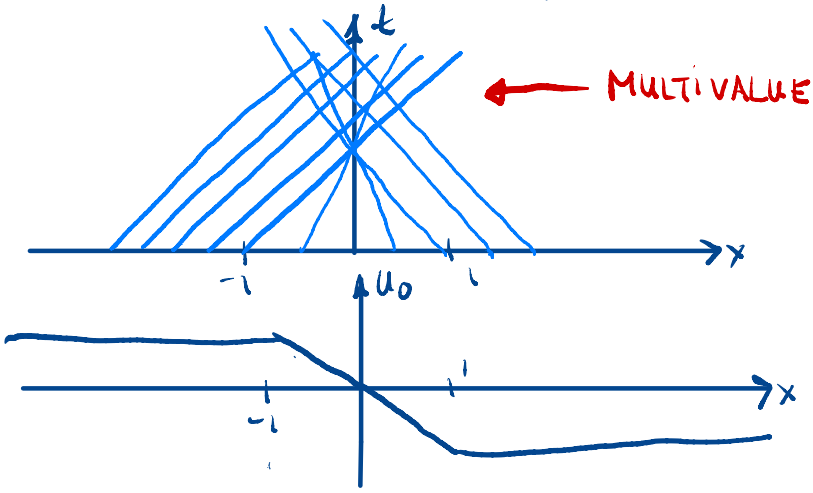
$$u_t = -\frac{x}{t^2} u_0'\left(\frac{x}{t}\right)$$

$$\left(\frac{1}{2} u_0^2\left(\frac{x}{t}\right)\right)_x = u_0\left(\frac{x}{t}\right) u_0'\left(\frac{x}{t}\right) \cdot \frac{1}{t}$$

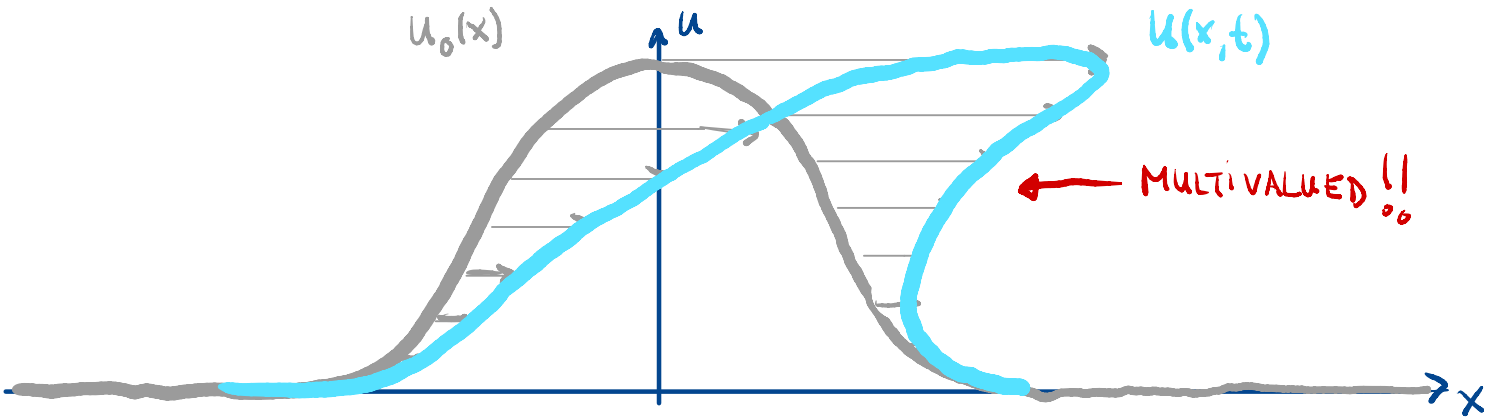
ok for $x/t \notin [-1,1]$ since $u_0' = 0$

for $x \in (-1,1)$ $u_0'(y) = 1$, so also ok.

What about $u_0(x) = \begin{cases} 1 & x \le -1 \\ -x & x \in [-1,1] \\ -1 & x > 1 \end{cases}$?



we have a type of "wave breaking"



if we want to have a solution after the breaking time, we must allow for discontinuous solutions.

Weak solutions

$\varphi(x,t)$ test function

$\varphi \in C_0^\infty(\mathbb{R}\times[0,\infty))$ $\varphi \in C^\infty$ with compact support

Definition:

$u = u(x,t)$ is a weak solution of $\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u_0 \end{cases}$

if

$$\int_0^\infty \int_{\mathbb{R}} u\,\varphi_t + f(u)\,\varphi_x \, dx\,dt + \int_{\mathbb{R}} u_0(x)\,\varphi(x,0)\,dx = 0$$

for all test functions φ.

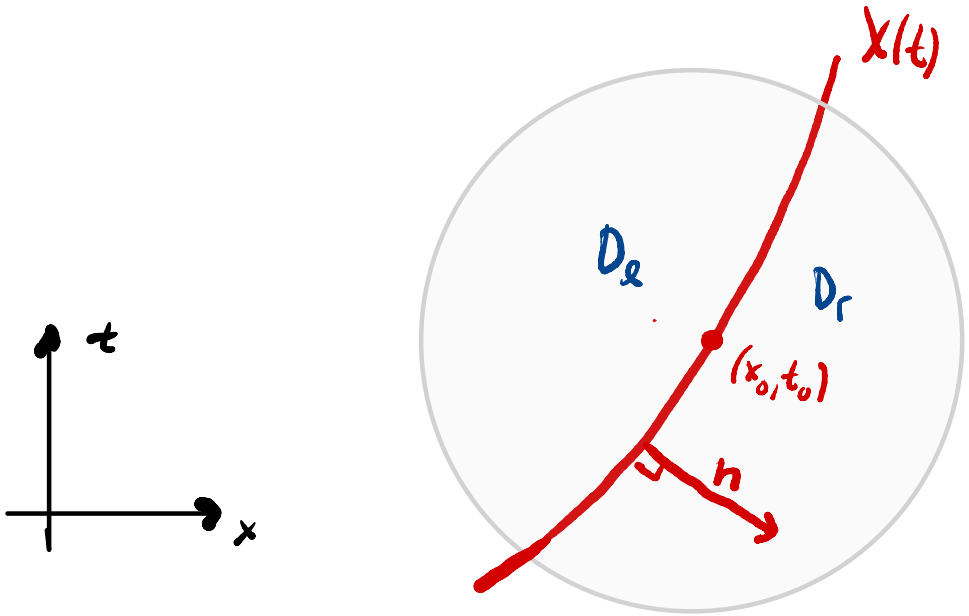
this makes sense if u is discontinuous.

classical solution $\Rightarrow$ weak solution

What kind of discontinuities can we have?

Simple case: Discontinuity at $x = X(t)$, $x_0 = X(t_0)$

u is differentiable on both sides of $X(t)$ near (x_0, t_0)



Support of φ

$D = D_l \cup D_r$

$$\lim_{\substack{x \to x_0 \\ x < x_0 \\ t \to t_0}} u(x,t) = u_l \qquad \lim_{\substack{x \to x_0 \\ x > x_0 \\ t \to t_0}} u(x,t) = u_r$$

u is a weak solution:

$$0 = \iint_D u\varphi_t + f(u)\varphi_x \, dx\,dt = \left(\iint_{D_l} + \iint_{D_r}\right) u\varphi_t + f(u)\varphi_x \, dx\,dt$$

integration by parts

$\varphi = 0$ on ∂D

$$= \int_X \varphi\,(f(u), u) \cdot n \, ds - \int_X \varphi\,(f(u), u) \cdot n \, ds$$

↑ left side of X ↑ right side of X

$n = (1, -x'(t))$

$$= \int_X \varphi \left(x'(t)(u_\ell - u_r) - (f(u_\ell) - f(u_r)) \right) ds$$

Since this is to hold for <u>all</u> test functions φ

$\Rightarrow$ $s = x'(t_0)$ satisfies

$$s = \frac{[\![f]\!]}{[\![u]\!]} \qquad [\![a]\!] := a_r - a_\ell$$

Rankine-Hugoniot condition $\leftrightarrow$ conservation

$$s(u_\ell - u_r) = f(u_\ell) - f(u_r)$$

An <u>example with many solutions</u>

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0 \qquad t > 0,\ x \in \mathbb{R}$$

$$u(x,0) = \begin{cases} -1 & x \le 0 \\ 1 & x > 0 \end{cases}$$

Claim $u(x,t) = u(x,0)$ is a weak solution.

<u>Check</u>:

$$\int_0^\infty \int_{\mathbb{R}} u \varphi_t \, dx\, dt = -\int_0^\infty \int_{-\infty}^0 \varphi_t \, dx\, dt + \int_0^\infty \int_0^\infty \varphi_t \, dx\, dt$$

$$= \int_{-\infty}^0 \varphi(x,0)\, dx - \int_0^\infty \varphi(x,0)\, dx = -\int_{\mathbb{R}} \varphi(x,0)\, u(x,0)\, dx$$

$$\int_0^\infty \int_{\mathbb{R}} f(u)\varphi_x \, dx\,dt = \int_0^\infty \int_{\mathbb{R}} \frac{1}{2}\varphi_x \, dx\,dt = 0$$

therefore

$$\int_0^\infty \int_{\mathbb{R}} \varphi_t u + \varphi_x f(u)\, dx\,dt + \int_{\mathbb{R}} \varphi(x,0)\,u(x,0)\,dx = 0$$

Set $u_1 = u$

Claim 2

$$u(x,t) = \begin{cases} x/t & |x| \le t \\ -1 & x < -t \\ 1 & x > t \end{cases}$$ is a weak solution

Check:

for $x > t$ $\quad u_t + f(u)_x = 0$ for $x < -t$ also.

for $x \in (-t,t)$:

$$u_t = -\frac{x}{t^2}, \quad u_x = \frac{1}{t} \Rightarrow u_t + f(u)_x = 0$$

Observe that u is continuous for $t > 0$, and satisfies $u_t + f(u)_x = 0$ almost everywhere.

Thus:

$$0 = \int_\varepsilon^\infty \int_{\mathbb{R}} \varphi(u_t + f(u)_x)\, dx\,dt = -\int_{\mathbb{R}^n} \varphi(x,\varepsilon)\,u(x,\varepsilon)\,dx - \int_\varepsilon^\infty \int_{\mathbb{R}} \varphi_t u + \varphi_x f(u)\, dx\,dt$$

for any positive ε.

We have that $\int_\varepsilon^\infty \varphi(x,\varepsilon)\,\varphi(\varepsilon,x)\,dx \longrightarrow \int_0^\infty \varphi(x,0)\,u(x,0)\,dx$

↑ Check!

Set $u_2 = u$.

For any positive c set $u_c = \begin{cases} u_1 & t<c \\ u_2(t-c) & t \geq c \end{cases}$

then u_c is also a weak solution!

This is not good, we must resolve the situation...

Vanishing viscosity

(V) $\quad u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u_{xx}, \quad \varepsilon > 0 \qquad u^\varepsilon(x,0) = u_0(x)$

Assume for the moment (we shall prove it later) that $u^\varepsilon \in C^\infty(\mathbb{R} \times (\delta, \infty))$ for any $\delta > 0$, and that $u^\varepsilon(x,\delta) \to u_0(x)$ in $L^1(\mathbb{R})$ as $\delta \to 0$.

How can we show that (V) has unique solutions?

Assume that f is Lipschitz continuous, let u & v be solutions and set $w = u - v$. Then

$$w_t + (f(u) - f(v))_x = \varepsilon w_{xx}$$

$$\boxed{w_t + (Aw)_x = \varepsilon w_{xx}} \qquad A = A(x,t) = \frac{f(u) - f(v)}{u - v}$$

Set $\eta(w) = |w| \qquad \eta'(w) = \operatorname{sgn}(w)$ if $w \neq 0$

Multiply with $\eta'(w)$ and use the chain rule

$$\eta(w)_t + A_x \eta(w) + A(w)\eta(w)_x = \eta'(w)\varepsilon w_{xx}$$

$$\eta(w)_t + (A\eta(w))_x = \varepsilon \eta(w)_{xx} - \varepsilon \eta''(w)(w_x)^2$$

← This does not really exist, formal

$$\eta(w)_x + (Aw)_x \leq \varepsilon \eta(w)_{xx}$$

← Can be proved anyway!

Integrate over $\mathbb{R}$ to get

$$\frac{d}{dt}\int_{\mathbb{R}} |w|\,dx \leq 0 \quad \Rightarrow \|w(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|w(\cdot,0)\|_{L^1(\mathbb{R})}.$$

Then (V) has unique solutions.

If we demand that

$$u = \lim_{\varepsilon \to 0} u^\varepsilon$$

maybe the uniqueness property will be inherited by u?

SPOILER: YES

Plan:

1. Prove that $u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}$ has a unique sol.
2. Prove that, with some extra condition, $u_t + f(u)_x = 0$ has unique solutions.
3. Prove that $\lim_{\varepsilon \to 0} u^\varepsilon = u$

$\lim_{\varepsilon \to 0} u^\varepsilon$: solves the viscous problem

u: solves the conservation law