Hamilton-Jacobi equations

$$(HJ) \quad \begin{cases} u_t + H(Du, x) = 0 & t>0 \quad x \in \mathbb{R}^n \\ u(x,0) = g(x) \end{cases}$$

$u : \mathbb{R}^n \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ unknown

$H : \mathbb{R}^n \times \mathbb{R}^n$ Hamiltonian

Connection to conservation laws

$p = Du \rightarrow p_t + H(p,x)_x = 0$ system of CL

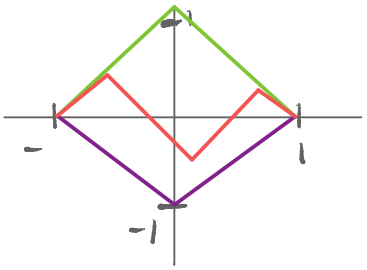
$n=1$

$p_t + H(p,x)_x = 0$ scalar conservation law,

entropy solution, L^1 well-poredness etc...

Example:

$$|u_x| = 1 \quad x \in (-1,1] \quad u(-1) = u(1) = 0$$



Many solutions!

For conservation laws $u_t + f(u)_x = 0 \longrightarrow$ Weak sol. $\iint u\varphi_t + f(u)\varphi_x \, dx\,dt = 0$

Move the derivatives to test function.

Cannot do the by IBP for HJ!

How to define a "weak solution":

$$(VS) \quad \begin{cases} u^\varepsilon_t + H(Du^\varepsilon, x) = \varepsilon \Delta u^\varepsilon, & t>0, \\ u^\varepsilon(x,0) = g(x) \end{cases}$$

Assume that $u^\varepsilon \longrightarrow u$ **uniformly in** $\mathbb{R}^n \times [0,\infty)$

Let v be a (test) function in $C^\infty(\mathbb{R}^n \times [0,\infty))$

If $(u-v)(x,t)$ has a **maximum** at (x_0,t_0)

then $(u^\varepsilon - v)$ has a maximum at (x_j,t_j) near (x_0,t_0) and

$$(x_j,t_j) \longrightarrow (x_0,t_0) \quad \text{as} \quad j \to \infty.$$

We have that

$$D(u^\varepsilon - v)(x_j,t_j) = 0 \quad \text{or} \quad Du^\varepsilon(x_j,t_j) = Dv(x_j,t_j)$$

$$\Delta(u^\varepsilon - v)(x_j,t_j) \leq 0 \quad \text{or} \quad \Delta u^\varepsilon(x_j,t_j) \leq \Delta v(x_j,t_j)$$

We calculate: (at (x_j,t_j))

$$\begin{aligned} v_t(x_j,t_j) + H(Dv(x_j,t_j),x_j) &= u^\varepsilon_t(x_j,t_j) + H(Du^\varepsilon(x_j,t_j),x_j) \\ &= \varepsilon \Delta u^\varepsilon(x_j,t_j) \\ &\leq \varepsilon \Delta v(x_j,t_j) \end{aligned}$$

Let now $j \to \infty$,

$$v_t(x_0,t_0) + H(Dv(x_0,t_0),x_0) \leq \varepsilon \Delta v(x_0,t_0)$$

Let now $\varepsilon \to 0$

$$\boxed{v_t(x_0,t_0) + H(Dv(x_0,t_0),x_0) \leq 0}$$

Definition of viscosity solution

A bounded, uniformly continuous (BUC) function u is called a viscosity solution of (HJ) if

a) $$u(x,0) = g(x),$$

and if for all test functions $v \in C^\infty(\mathbb{R}^n, [0,\infty))$,

1 { if $u - v$ has a local max. at (x_0,t_0) then
$$v_t(x_0,t_0) + H(Dv(x_0,t_0), x_0) \leq 0,$$

2 { if $u - v$ has a local min. at (x_0,t_0) then
$$v_t(x_0,t_0) + H(Dv(x_0,t_0), x_0) \geq 0.$$

1. subsolution 2. supersolution.

Exercise: Classical solution $\Rightarrow$ viscosity solution.

Lemma: If $u: \mathbb{R}^n \to \mathbb{R}$ is continuous, and differentiable at x_0, then there is a function $v \in C^1(\mathbb{R}^n)$ such that $v(x_0) = u(x_0)$ and $(u - v)$ has a strict local max. at x_0

Proof:

We can assume that $x_0 = 0$, $u(0) = Du(0) = 0$

Since u is continuous and differentiable at 0, there is a $p_1(x)$ such that

$$u(x) = |x| p_1(x) \qquad p_1 : \text{continuous}$$

Let

$$p_2 = \max_{B(0,r)} p_1(x)$$

p_2 is continuous and $p_2(r) \geq 0$, nondecreasing.