Definition of viscosity solution

A bounded, uniformly continuous (BUC) function u is called a viscosity solution of (HJ) if

a) $$u(x,0) = g(x),$$

and if for all test functions $v \in C^\infty(\mathbb{R}^n, [0,\infty))$,

1 { if $u - v$ has a local max. at (x_0,t_0) then
$$v_t(x_0,t_0) + H(Dv(x_0,t_0), x_0) \leq 0,$$

2 { if $u - v$ has a local min. at (x_0,t_0) then
$$v_t(x_0,t_0) + H(Dv(x_0,t_0), x_0) \geq 0.$$

1. subsolution 2. supersolution.

Exercise: Classical solution $\Rightarrow$ viscosity solution.

Lemma: If $u: \mathbb{R}^n \to \mathbb{R}$ is continuous, and differentiable at x_0, then there is a function $v \in C^1(\mathbb{R}^n)$ such that $v(x_0) = u(x_0)$ and $(u - v)$ has a strict local max. at x_0

Proof:

We can assume that $x_0 = 0$, $u(0) = Du(0) = 0$

Since u is continuous and differentiable at 0, there is a $p_1(x)$ such that

$$u(x) = |x| p_1(x) \qquad p_1 : \text{continuous}$$

Let

$$p_2 = \max_{B(0,r)} p_1(x)$$

p_2 is continuous and $p_2(r) \geq 0$, nondecreasing.

Define $v(x) = \int_{|x|}^{2|x|} p_2(r)\,dr + |x|^2$ $\quad v \in C^1(\mathbb{R}^n)$

$v(0) = 0$, $\quad Dv = \frac{x}{|x|}\left(2p_2(|2x|) - p_2(|x|) - 2|x|\right)$

$Dv(0) = 0.$ $\quad (x \to 0)$

$x \neq 0$, $\quad (u-v)(x) = |x|\,p_1(x) - \int_{|x|}^{2|x|} p_2(r)\,dr - |x|^2$

$\leq |x|\,p_2(|x|) - \int_{|x|}^{2|x|} p_2(r)\,dr - |x|^2 \leq -|x|^2$

(≤ 0)

< 0 for $x \neq 0$. ∎

Theorem: u viscosity solution. If u is differentiable at (x_0, t_0) then

$$u_t(x_0,t_0) + H(Du(x_0,t_0), x_0) = 0$$

Proof: By the lemma, there exists a C^1 function $v = v(x,t)$ s.t. $u - v$ has a strict minimum at (x_0,t_0), set $v^\varepsilon = \omega_\varepsilon * v$ (mollifier)

Then:

$$\left.\begin{array}{l} v^\varepsilon \to v \\ v^\varepsilon_t \to v_t \\ Dv^\varepsilon \to Dv \end{array}\right\} \text{ as } \varepsilon \to 0 \text{ near } (x_0,t_0), \text{ uniformly.}$$

Then $u - v^\varepsilon$ has a maximum at $(x^\varepsilon, t^\varepsilon)$ near (x_0,t_0).

Since u is a viscosity solution

$$v^\varepsilon_t(x^\varepsilon,t^\varepsilon) + H(Dv^\varepsilon(x^\varepsilon,t^\varepsilon), x^\varepsilon) \leq 0$$

Let $\varepsilon \to 0$

$$0 \geq v_t(x_0,t_0) + H(Dv(x_0,t_0), x_0) = u_t(x_0,t_0) + H(Du(x_0,t_0), x_0).$$

Using the lemma on $-u$, find a $v \in C^1$ such that $u - v$ has a strict minimum at (x_0,t_0) to get

$$0 \geq u_t(x_0,t_0) + H(Du(x_0,t_0), x_0).$$

We assume Lip continuity of H

$$|H(p,x) - H(q,x)| \leq C|p-q|$$
$$|H(p,x) - H(p,y)| \leq C(1+|p|)|x-y|$$

Theorem: There exists at most one viscosity solution of (HJ).

Idea of proof: Assume that $u, \tilde{u}$ are viscosity solns., and that $u - \tilde{u}$ has a positive max. at (x_0,t_0).

Double the variables! Choose test function

$$v(x,y,t,s) = u(x,t) - \tilde{u}(y,s) - \lambda(t-s) - \frac{1}{\varepsilon^2}\left(|x-y|^2 + (t-s)^2\right) - \varepsilon(|x|^2 + |y|^2)$$

($\lambda(t-s)$: this is a constant in (x,t)!)

(v: has max at (x_0,y_0,t_0,s_0))

($\frac{1}{\varepsilon^2}(|x-y|^2 + (t-s)^2)$: to make $(x^\varepsilon, t^\varepsilon) \approx (x_0,t_0)$)

($\varepsilon(|x|^2 + |y|^2)$: To ensure existence of max)

Use that u & $\tilde{u}$ are viscosity solns. to get a contradiction.