Existence of solutions via control theory.

CONTROL PROBLEM

We wish to control (by tuning α), solutions of the ODE

$$\begin{cases} x'(s) = f(x(s), \alpha(s)) & t < s < T \\ x(t) = x \end{cases} \qquad f(\mathbb{R}^n \times A) \to \mathbb{R}^n \text{ Lipschitz}$$

$$\mathcal{A} = \{ \alpha : [0,T] \to A \,,\ \alpha \text{ measurable} \} \leftarrow \text{control set.}$$

For each $\alpha \in \mathcal{A}$ there exists a unique solution $x(s)$.

We want to choose the control such that the cost is minimal.

$$C_{x,t} = \int_t^T r(x(s), \alpha(s))\, ds + g(x(T)) \leftarrow \text{Total cost.}$$

r = running cost. g = Terminal cost.

r and g are bounded and Lipschitz continuous.

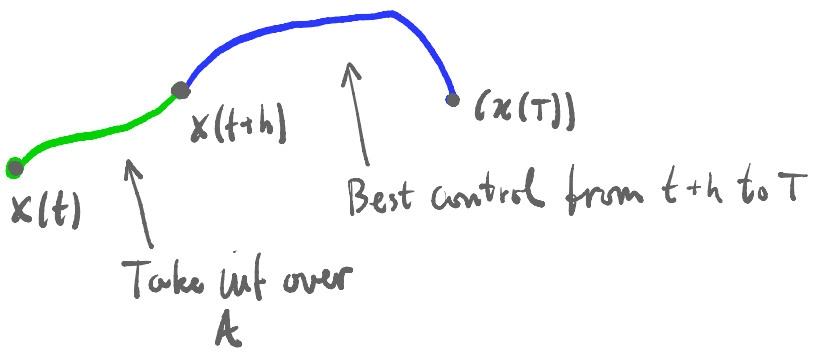
The value function

$$u(x,t) = \inf_{\alpha \in \mathcal{A}} \{ C_{x,t}(\alpha(\cdot)) \}.$$

DYNAMIC PROGRAMMING

Theorem: If $t + h \leq T$ we have

$$u(x,t) = \inf_{A} \left[\int_t^{t+h} r(x(s), \alpha(s))\, ds + u(x(t+h), t+h) \right]$$



Proof: Choose any control α_1, α_2

$$\begin{cases} x_1'(s) = f(x_1(s), \alpha_1(s)) & t < s < t+h \\ x_1(t) = x \end{cases}$$

$$\begin{cases} x_2'(s) = f(x_2(s), \alpha_2(s)) & t+h < s < T \\ \alpha_2(t+h) = x_1(t+h) \end{cases}$$

Set $\alpha_3(s) = \alpha_1(s)$ for $t < s < t+h$, and $\alpha_3(s) = \alpha_2(s)$ $t+h < s < T$.

$$u(x,t) \le C_{x_3,t}(\alpha_3) = \int_t^T r(x_3(s), \alpha_3(s))\,ds + g(x_3(T))$$

$$= \int_t^{t+h} r(x_1(s), \alpha_1(s))\,ds + \int_{t+h}^T r(x_2(s), \alpha_2(s))\,ds + g(x_2(T))$$

Choose α_2 to be nearly optimal

$$\le \int_t^{t+h} r(x_1(s), \alpha_1(s))\,ds + u(x_1(t+h), t+h) + \varepsilon$$

take int over α_1

$$u(x,t) \le \inf_{\alpha_1}\left[\int_t^{t+h} r(x(s), \alpha(s))\,ds + u[x(t+h), t+h]\right] + \varepsilon$$

Now choose α_4 so that

$$\begin{cases} x_4'(s) = f(x_4(s), \alpha_4(s)) & t < s < T \\ x_4(t) = x \end{cases}$$

$$u(x,t) + \varepsilon \ge \int_t^T r(x_4(s), \alpha_4(s))\,ds + g(x_4(T))$$

Next

$$u(x_4(t+h), t+h) \le \int_{t+h}^T r(x_4(s), \alpha_4(s))\,ds + g(x_4(T))$$

$$u(x,t)+\varepsilon \geq \int_t^{t+h} r(x_y(s),\alpha_y(s))\,ds + \int_{t+h}^{T} r(x_y(s),\alpha_y(s))\,ds + g(x_y(T))$$

$$\geq \inf_A \left[\int_t^{t+h} r(x(s),\alpha(s))\,ds + u(x(t+h),t+h) \right]$$ ∎

Estimates

We have assumed that

$$|r(x,a)| \leq C \qquad |g(x)| \leq C.$$

and therefore

$$|u(x,t)| \leq \int_t^T |r(x(s),\alpha(s))|\,ds + |g(x(T))|$$

$$\leq C(T-t) + C \quad \text{bounded}$$

Lipschitz continuity

Fix $x, \hat{x}$ and $0 \leq t \leq T$. Choose $\hat{\alpha}$ s.t

$$\begin{cases} \hat{x}'(s) = f(\hat{x}(s),\hat{\alpha}(s))\,ds \\ \hat{x}(t) = \hat{x} \end{cases}, \quad u(\hat{x},t)+\varepsilon \geq \int_t^T r(\hat{x}(s),\hat{\alpha}(s))\,ds + g(\hat{x}(T))$$

Then let $x(s)$ solve

$$\begin{cases} x'(s) = f(x(s),\hat{\alpha}(s)) & t \leq s \leq T \\ x(t) = x \end{cases}$$

$\hat{\alpha}$ is not the optimal control at (x,t),

$$u(x,t) - u(\hat{x},t) \leq \int_0^T r(x(s),\hat{\alpha}(s))\,ds + g(x(T))$$

$$- \int_0^T r(\hat{x}(s),\hat{\alpha}(s))\,ds - g(\hat{x}(T)) + \varepsilon$$

$$\leq \int_0^T C|x(s)-\hat{x}(s)|\,ds + C|x(T)-\hat{x}(T)| + \varepsilon$$

$$\leq C|x-\hat{x}| + \varepsilon$$ ← Have used Gronwall's inequality

Can use the same argument on $u(\hat{x},t) - u(x,t)$.

Hence
$$|u(x,t)-u(\hat{x},t)| \leq C|x-\hat{x}|.$$

Estimating $|u(x,t)-u(x,\hat{t})|$

Choose $0 \leq t \leq \hat{t} \leq T$ and a nearly optimal control α for x, s.t.

$$u(x,t)+\varepsilon \geq \int_t^T r(x(s),\alpha(s))\,ds + g(x(T))$$

Define $\hat{\alpha}(s) = \alpha(s+t-\hat{t})$ for $\hat{t} \leq s \leq T$

$\hat{\alpha}$ not optimal for $\hat{x}$.

$$\begin{cases} \hat{x}'(s) = f(\hat{x}(s), \hat{\alpha}(s))\,ds \\ \hat{x}(\hat{t}) = x. \end{cases}$$

Note $\hat{x}(s) = x(s+t-\hat{t})$

$$u(x,\hat{t}) - u(x,t) = \int_{\hat{t}}^T r(\hat{x}(s),\hat{\alpha}(s))\,ds + g(\hat{x}(T)) - \int_t^T r(x(s),\alpha(s))\,ds - g(x(T)) + \varepsilon$$

$$= \int_t^{T-\hat{t}+t} r(x(s),\alpha(s))\,ds - \int_t^T r(x(s),\alpha(s))\,ds + g(x(T-\hat{t})) - x(T)| + \varepsilon$$

$$\leq \int_{T-\hat{t}+t}^T C'|r(x(s),\alpha(s))|\,ds + C|g(x(T-\hat{t}+t)) - x(T)|$$

$$\leq C|t-\hat{t}| + \varepsilon.$$

Can use the "same" for $u(x,t) - u(x,\hat{t})$

A differential equation for u

Theorem.

u is the unique viscosity solution of the backward Hamilton-Jacobi-Bellman equation

$$\begin{cases} u_t + H(Du, x) = 0 & 0 \le t < T \\ u(x, T) = g(x) \end{cases}$$

$$H(p, x) = \min_{a \in A} \{ f(x, a) \cdot p + r(x, a) \}$$

Example

$n = 1$ $\quad f(x, a) = x + a$, $\quad r(x, a) = \frac{1}{2}x^2 + \frac{1}{2}a^2$

$g(x) = |x|$

Finding H.

$$\min_a \{ xp + ap + \tfrac{1}{2}x^2 + \tfrac{1}{2}a^2 \}$$

$$= \tfrac{1}{2}x^2 + px - \tfrac{1}{2}p^2$$

HJB:

$$u_t + \tfrac{1}{2}x^2 + xu_x - \tfrac{1}{2}(u_x)^2 = 0 \quad 0 \le t < T$$

$$u(x, T) = |x|$$

HJB is a terminal value problem, must change the definition of a viscosity solution accordingly:

$u - v$ has max at (x_0, t_0)

$\Downarrow$

$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \ge 0$

supersolution

$u - v$ has min at (x_0, t_0)

$\Downarrow$

$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \le 0$

subsolution