# A differential equation for $u$

**Theorem.**

$u$ is the unique viscosity solution of the backward Hamilton-Jacobi-Bellman equation

$$\begin{cases} u_t + H(Du, x) = 0 & 0 \le t < T \\ u(x, T) = g(x) \end{cases}$$

$$H(p, x) = \min_{a \in A} \{ f(x, a) \cdot p + r(x, a) \}$$

**Example**

$n = 1$ $\quad f(x, a) = x + a$, $\quad r(x, a) = \frac{1}{2}x^2 + \frac{1}{2}a^2$

$g(x) = |x|$

Finding $H$.

$$\min_a \{ xp + ap + \tfrac{1}{2}x^2 + \tfrac{1}{2}a^2 \}$$

$$= \tfrac{1}{2}x^2 + px - \tfrac{1}{2}p^2$$

HJB:

$$u_t + \tfrac{1}{2}x^2 + xu_x - \tfrac{1}{2}(u_x)^2 = 0 \quad 0 \le t < T$$

$$u(x, T) = |x|$$

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HJB is a terminal value problem, must change the definition of a viscosity solution accordingly:

$u - v$ has max at $(x_0, t_0)$

$\Downarrow$

$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \ge 0$

supersolution

$u - v$ has min at $(x_0, t_0)$

$\Downarrow$

$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \le 0$

subsolution

**Exercise:** Check that $w(x,t) := u(x, T-t)$ is a viscosity solution of

$$\begin{cases} u_t - H(Du, x) = 0 & 0 < t \leq T \\ u(x,0) = g(x) \end{cases}$$

We prove that the value function is a viscosity solution of HJB.

Correct terminal value:

$$u(x,T) = \inf_{\alpha \in A} C_{x,T}[\alpha(\cdot)] = g(x) \quad \checkmark$$

We have that $u$ is bounded and Lipschiz continuous.

Assume that

$u - v$ has a max at $(x_0, t_0)$

we must prove that

$$v_t(x_0,t_0) + \min_a \left( f(x_0,a) \cdot Dv(x_0,t_0) + r(x_0,a) \right) \geq 0.$$

Proof by Contradiction. Suppose **not**. Since $v$ is $C^1$, then for all $(x,t)$ such that $|x - x_0| + |t - t_0| < \delta$, we have that

$$v_t(x,t) + f(x,a) \cdot Dv(x,t) - r(x,a) \leq -\theta < 0$$

for some $a \in A$

By possibly taking a smaller $\delta$, we can assume that

$$(u-v)(x,t) \leq (u-v)(x_0,t_0) \quad \text{for all } (x,t) \text{ near } (x_0,t_0)$$

Consider the constant control $\alpha(s) \equiv a$

$$\begin{cases} x'(s) = f(x(s), a) & t_0 \leq t \leq T \\ x(t_0) = x_0 \end{cases}$$

Choose $0 < h < \delta$ so small that $|x(s) - x_0| < \delta$

for $t_0 < s < t_0+h$. Using $x(s)$ in the assumption

$$v_t(x(s),s) + f(x(s),a)\cdot Dv(x(s),s) + r(x(s),a) \le -\theta$$

We also have (since $u-v$ has max at $(x_0,t_0)$

$$u(x(t_0+h),t_0+h) - u(x_0,t_0) \le v(x(t_0+h),t_0+h) - v(x_0,t_0)$$

$$= \int_{t_0}^{t_0+h} \frac{d}{ds} v(x(s),s)\,ds = \int_{t_0}^{t_0+h} v_t(x(s),s) + Dv(x(s),s)\cdot x'(s)\,ds$$

$$= \int_{t_0}^{t_0+h} v_t(x(s),s) + f(x(s),a)\cdot Dv(x(s),s)\,ds \ge u(x(t_0+h),t_0+h) - u(x_0,t_0)$$

Also (by DP)

$$u(x_0,t_0) \le \int_{t_0}^{t_0+h} r(x(s),a)\,ds + u(x(t_0+h),t_0+h)$$

or $$u(x(t_0+h),t_0+h) - u(x_0,t_0) \ge -\int_{t_0}^{t_0+h} r(x(s),a)\,ds$$

$$\Rightarrow 0 \le \int_{t_0}^{t_0+h} v_t(x(s),s) + f(x(s),a)\cdot Dv(x(s),s) + r(x(s),a)\,ds \le -\theta h.$$

contradiction!

Assume that $(u-v)$ has a minimum at $(x_0,t_0)$, then we must show that $v_t(x_0,t_0) + f(x_0,a)\cdot Dv(x_0,t_0) + r(x_0,a) \le 0$

Suppose not, then for $(x,t)$ near $(x_0,t_0)$ and all $a \in A$

$$v_t(x,t) + f(x,a)\cdot Dv(x,a) + r(x,a) \ge \theta > 0.$$

Let $x(s)$ be some control $h$ small so that $(x(s),s)$ near $(x_0,t_0)$ for $t_0 \le s \le t_0+h$.

Let $x(s)$ solve

$$\begin{cases} x'(s) = f(x(s), \alpha(s)) & t_0 \le s \le t_0 + h \\ x(t_0) = x_0 \end{cases}$$

Local minimum $\Rightarrow$

$$u(x(t_0+h), t_0+h) - u(x_0, t_0) \ge v(x(t_0+h), t_0+h) - v(x_0, t_0)$$

$$= \int_{t_0}^{t_0+h} v_t(x(s), s) + f(x(s), \alpha(s)) \cdot Dv(x(s), s)\, ds$$

Dynamic Programming + choosing a "good" control $\alpha$

$$u(x_0, t_0) \ge \int_{t_0}^{t_0+h} r(x(s), \alpha(s))\, ds + u(x(t_0+h), t_0+h) - \frac{\theta h}{2}$$

$\Downarrow$

$$\frac{\theta h}{2} \ge u(x(t_0+h), t_0+h) - u(x_0, t_0) + \int_{t_0}^{t_0+h} r(x(s), s)\, ds$$

$$\ge \int_{t_0}^{t_0+h} v_t(x(s), s) + f(x(s), \alpha(s)) \cdot Dv(x(s), s) + r(x(s), \alpha(s))\, ds$$

$$\ge \theta h$$ Contradiction!

The value function $u$ is a viscosity solution of HJB