Hamilton-Jacobi equations vs Conservation laws

Let $n=1$ and $H(p,x) = H(x)$.

Formally:

$$\text{HJ} \quad \begin{cases} u_t + H(u_x) = 0 & t>0 \\ u(x,0) = g(x) \end{cases} \qquad x \in \mathbb{R}$$

Apply $\frac{\partial}{\partial x}$ and define $p = u_x$.

$$\begin{cases} u_{xt} + H(u_x)_x = 0 \\ u_x(x,0) = g'(x) \end{cases} = \begin{cases} p_t + H(p)_x = 0 \\ p(x,0) = p_0(x) = g'(x) \end{cases} \qquad \text{CL}$$

It is true that if u is a viscosity solution of HJ then $p = u_x$ is an entropy solution of CL.

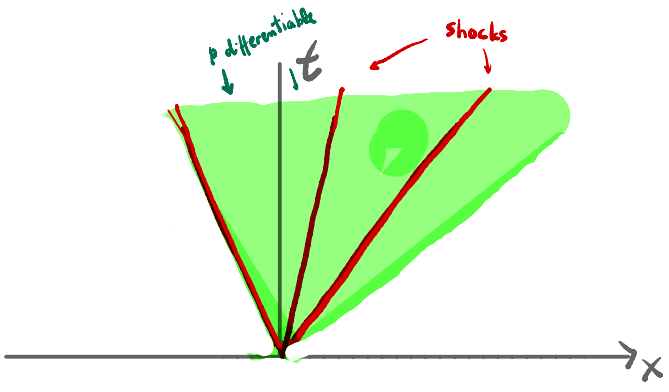
And vice versa, if p is an entropy solution, then $u = \int_{-\infty}^{x} p\,dx$ is a viscosity solution.

We shall show this for the Riemann problem.

Riemann problem for the conservation law

$$p_t + H(p)_x = 0 \qquad t>0$$

$$p(x,0) = \begin{cases} p_\ell & x<0 \\ p_r & x>0 \end{cases}$$



$$p(x,t) = P\left(\frac{x}{t}\right)$$

$$P(\xi) = (H_c')^{-1}(\xi)$$ lower

or $$P(\xi) = (H_n')^{-1}(\xi)$$ upper

concave envelopes

σ speed of shock satisfies RH-condition

$$\sigma(p_r - p_\ell) = H(p_r) - H(p_\ell)$$ p_ℓ, p_r = left/right limits.

Entropy condition.

$p_\ell < p_r$

$$H(p_\ell) + \sigma(p - p_\ell) \le H(p)$$

for all $p \in [p_\ell, p_r]$

$p_\ell > p_r$

$$H(p_\ell) + \sigma(p - p_\ell) \ge H(p)$$

for all $p \in [p_r, p_\ell]$

Since $p(x,t) = P\left(\frac{x}{t}\right)$:

$$p_x = \frac{1}{t}\dot{P}$$

$$p_t = -\frac{x}{t^2}\dot{P} = -\frac{x}{t}p_x$$

$$p_t = -\frac{x}{t}p_x$$

The Riemann problem for HJ

$$u_t + H(u_x) = 0 \qquad t > 0$$

$$u(x,0) = \begin{cases} p_\ell x & x < 0 \\ p_r x & x \ge 0 \end{cases}$$

Theorem: Let $p = p(x,t)$ be the entropy solution of the conservation law. Define

$$u(x,t) = xp(x,t) - tH(p(x,t))$$

Then u is the viscosity solution of HJ.

We first show that u is BUC, if p is differentiable at (x,t)

$$u_x(x,t) = p(x,t) + xp_x(x,t) - tH(p(x,t))_x$$

$$= p(x,t) - tp_t(x,t) - tH(p(x,t))_x$$

$$= p(x,t) - t(p_t + H(p)_x) = p(x,t)$$

If p is <u>not</u> differentiable at (x,t) then it has right and left limits p_ℓ, p_r at (x,t). Furthermore $x = \sigma t$ where σ satisfies:

$$\sigma = \frac{H(p_r) - H(p_\ell)}{p_r - p_\ell} \qquad \text{if } p \text{ has a jump}$$

$$\sigma = H'(p_{\ell,r}) \qquad \text{otherwise.}$$

We check that u is continuous at (x,t).

$$u_\ell := \lim_{y \to x-} u(y,t) = xp_\ell - tH(p_\ell) = (\sigma p_\ell + H(p_\ell))t$$

$$u_r := \lim_{y \to x+} u(y,t) = xp_r - tH(p_r) = (\sigma p_r + H(p_r))t$$

The RH-condition implies that $u_r = u_\ell$.

Hence u is BUC.

We must check if u is a viscosity solution. If u is differentiable at (x_0,t_0) and $(u-v)$ has a max/min at (x_0,t_0) then

$u_x(x_0,t_0) = v_x(x_0,t_0)$ and $v_t(x_0,t_0) = u(x_0,t_0)$ so that

$$v_t(x_0,t_0) + H(v_x(x_0,t_0)) = 0$$

It remains to check the case where (x_0,t_0) is on a shock.

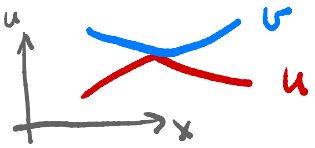
In this case $x_0 = \sigma t_0$ where σ satisfies RH.

$$\sigma(p_r - p_\ell) = H(p_r) - H(p_\ell)$$

where

$$p_\ell = \lim_{y \to x_0^-} p(y,t_0) \qquad p_r = \lim_{y \to x_0^+} p(y,t_0)$$

Assume that $(u-v)$ has a local max at (x_0,t_0)

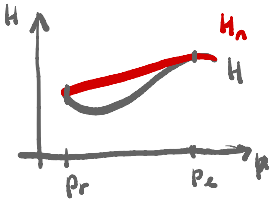


In this case $p_\ell = \lim_{y \to x_0^-} u_x(y,t_0)$

and $p_r := \lim_{y \to x_0^+} u_x(y,t_0)$ satisfy

$$p_\ell \geq v_x(x_0,t_0) \geq p_r.$$

In particular $p_\ell \geq p_r$ and the Riemann problem for CL is solved by using the upper concave envelope $H_\cap$



$$H(p) \leq H(p_\ell) + \sigma(p - p_\ell) \quad \text{for all } p \in [p_r, p_\ell] \quad *$$

Choose (x,t) such that $\sigma(t-t_0) = x - x_0$, $t < t_0$, (x,t) near (x_0,t_0)

$$\frac{u(x,t) - u(x_0,t_0)}{t-t_0} \geq \frac{v(x,t) - v(x_0,t_0)}{t-t_0} \qquad t - t_0 < 0$$

$$\frac{(x-x_0)p_\ell - (t-t_0)H(p_\ell)}{t-t_0} \geq \frac{v(x,t) - v(x_0,t_0)}{t-t_0} \qquad \text{formula for } u$$

$$t \to t_0^-$$

$$\sigma p_\ell - H(p_\ell) \geq v_t(x_0,t_0) + \sigma v_x(x_0,t_0) \qquad \text{chain rule}$$

$$\sigma(p_\ell - v_x(x_0,t_0)) + H(v_x(x_0,t)) - H(p_\ell) \qquad \text{add \& subtract } H(v_x)$$

$$\geq v_t(x_0,t_0) + H(v_x(x_0,t_0))$$

$$0 \geq v_t(x_0,t_0) + H(v_x(x_0,t_0))$$

u is a subsolution, use similar arguments to show that u is a supersolution ∎

Do this!

Examples:

$$H(p) = \frac{1}{2}p^2$$

$$p_t + H(p)_x = 0$$

$$p(x,0) = \begin{cases} 0 & x<0 \\ 1 & x\geq 0 \end{cases} \Rightarrow p(x,t) = \begin{cases} 0 & x<0 \\ x/t & 0\leq x<t \\ 1 & x\geq t \end{cases}$$

$$u_t + H(u_x) = 0$$

$$u(x,0) = \begin{cases} 0 & x\leq 0 \\ x & x\geq 0 \end{cases}$$

Solution

$$u(x,t) = xp(x,t) - tH(p(x,t)) = \begin{cases} 0 & x<0 \\ \frac{1}{2}\frac{x^2}{t} & 0\leq x\leq t \\ x-\frac{t}{2} & t\leq x \end{cases}$$

$$p(x,0) = \begin{cases} 1 & x<0 \\ 0 & x>0 \end{cases} \qquad u(x,0) = \begin{cases} x & x<0 \\ 0 & x>0 \end{cases}$$

$$\Rightarrow$$

$$p(x,t) = \begin{cases} 1 & x < t/2 \\ 0 & x\geq t/2 \end{cases} \qquad u(x,t) = \begin{cases} x-t/2 & x<t/2 \\ 0 & x\geq t/2 \end{cases}$$