The Laplace equation

$$\Delta u(x) = 0, \quad x \in \mathbb{R}^n$$

We try to find "some kind of formula".

Assume that $u(x) = v(r)$

$$r = |x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

$$r_{x_i} = \frac{\partial}{\partial x_i}(x_1^2 + \dots + x_i^2 + \dots + x_n^2)^{1/2} = \frac{x_i}{r}$$

$$r_{x_i x_i} = \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = \frac{r - x_i \frac{x_i}{r}}{r^2} = (r - \frac{x_i^2}{r})/r^2$$

$$u_{x_i x_i} = v(r)_{x_i x_i} = (v'(r) r_{x_i})_{x_i}$$

$$= v''(r) r_{x_i}^2 + v'(r) r_{x_i x_i}$$

$$= v''(r) \frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \sum_{i=1}^{n} v''(r) \frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{1}{r}\frac{x_i^2}{r^2}\right)$$

$$= v''(r) + v'(r)\left(\frac{n-1}{r}\right) = 0.$$

If $r \neq 0$, $v \neq 0$

$$\frac{v''}{v'} = -\frac{n-1}{r} \qquad n \geq 2.$$

$$\Rightarrow \quad \frac{d}{dr}\log(v') = -(n-1)\frac{d}{dr}\log(r) = \log(r^{1-n})$$

$$\Rightarrow \quad v' = C\, r^{1-n}$$

$$v(r) = \begin{cases} C\log(r) & n=2 \\ C\, r^{2-n} & n \geq 3. \end{cases}$$

Define the fundamental solution:

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|) & n=2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

$\alpha(n)$: volume of the sphere

Observe:

$$\lim_{x \to 0} \Phi(x) = \infty$$

$$|D\Phi(x)| \leq C \frac{1}{|x|^{n-1}} \quad \leftarrow \text{ locally integrable}$$

$$|D^2\Phi(x)| \leq C \frac{1}{|x|^{n}} \quad \leftarrow \text{ not locally integrable near } 0$$

Solving the Poisson equation on $\mathbb{R}^n$

(P) $\quad -\Delta u = f \quad \text{in } \mathbb{R}^n$

Assume that:

i) f has compact support

($f(x) \equiv 0$ if $|x| > R$)

ii) $f \in C^2(\mathbb{R}^n)$

(S) $\quad u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$

Theorem: a) $u \in C^2(\mathbb{R}^n)$

b) u solves (P)

Proof of a):

$$u(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\, dy = \int_{\mathbb{R}^n} K(y) f(x-y)\, dy$$

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \underbrace{\frac{1}{h}\left(f(x+he_i-y) - f(x-y)\right)}_{\downarrow h \to 0} dy$$

$\downarrow h \to 0$

$$u_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(y)\, f_{x_i}(x-y)\, dy$$

same method for the second order derivatives. ∎

Proof of b):

$$\Delta u(x) = \int_{\mathbb{R}^n} \Phi(y)\, \Delta_x f(x-y)\, dy$$

$$= \underbrace{\int_{B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy}_{I_\varepsilon} + \underbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy}_{J_\varepsilon}$$

we want to make this small.

we have

$n=2$

$$\int_{B(0,\varepsilon)} |\Phi(y)|\, dy = C' \int_{B(0,\varepsilon)} |\log(r)|\, dy = C \int_0^\varepsilon r \log(r)\, dr \leq C\varepsilon^2 \log(\varepsilon)$$

polar coordinates

$n \geq 3$

$$\int_{B(0,\varepsilon)} |\Phi(y)|\, dy = C' \int_{B(0,\varepsilon)} \frac{1}{|x|^{n-2}}\, dx = C' \int_0^\varepsilon \frac{1}{r^{n-2}} r^{n-1}\, dr = C\varepsilon^2$$

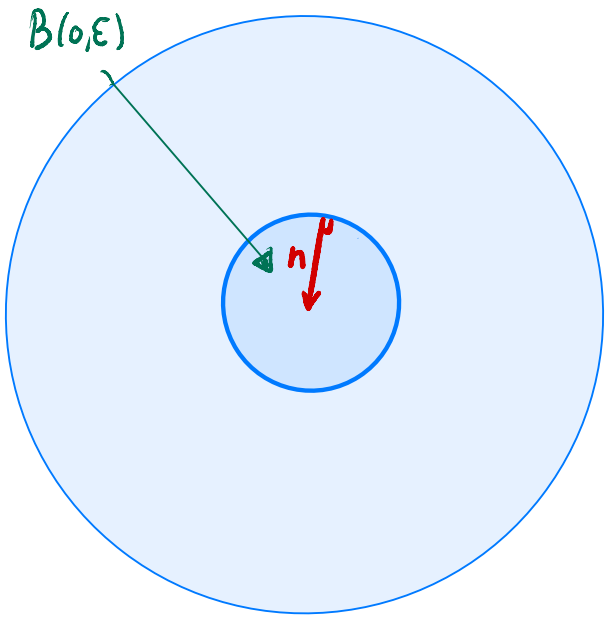
Therefore

$$|I_\varepsilon| = \left| \int_{B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy \right| \leq \int_{B(0,\varepsilon)} |\Phi(y)|\, |\Delta_x f(x-y)|\, dy$$

$$\leq C' \|\Delta_x f\|_{L^\infty(\mathbb{R}^n)} \begin{cases} \varepsilon^2 \log(\varepsilon) & n=2 \\ \varepsilon^2 & n \geq 3 \end{cases} \longrightarrow 0 \text{ as } \varepsilon \to 0$$

Estimate for J_ε

Support of f is inside $\square$



$$\Delta_x f(x-y) = \Delta_y f(x-y)$$

$$\Downarrow$$

$$J_\varepsilon = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\, \Delta_y f(x-y)\, dy$$

integration by parts

$$= -\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} D\Phi(y) \cdot D_y f(x-y)\, dy \longleftarrow K_\varepsilon$$

$$+ \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial n}(x-y)\, dS(y) \longleftarrow L_\varepsilon$$

$$y \in \partial B(0,\varepsilon) \quad \Phi(y) = C' \begin{cases} \log(\varepsilon) & n=2 \\ \left(\frac{1}{\varepsilon}\right)^{n-2} & n \geq 3 \end{cases}$$

$$|L_\varepsilon| \leq C |\Phi(\varepsilon)|\, \|Df\|_{L^\infty(\mathbb{R}^n)} (\text{area of } \partial B(0,\varepsilon))$$

$$\leq C' \varepsilon^{n-1} \begin{cases} \log(\varepsilon) & n=2 \\ \frac{1}{\varepsilon^{n-2}} & n \geq 3 \end{cases} \longrightarrow 0 \text{ as } \varepsilon \longrightarrow \varepsilon$$

integration by parts

$$K_\varepsilon = \underbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta\Phi(y)\, f(x-y)\, dy}_{=0} - \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial n}(y)\, f(x-y)\, dy$$

since $\Delta\Phi(y) = 0$ here!

Recall: $n=2$ $\Phi(y) = -\frac{1}{2\pi}\log(|y|)$

$$D\Phi(y) = -\frac{1}{2\pi}\frac{1}{|y|}\frac{y}{|y|} = -\frac{1}{2\alpha(2)}\frac{y}{|y|^2}$$

$n \geq 3$ $\Phi(y) = \frac{1}{n(n-2)\alpha(n)}\frac{1}{|y|^{n-2}}$

$$D\Phi(y) = \frac{-(n-2)}{n(n-2)\alpha(n)}\frac{1}{|y|^{n-1}}\frac{y}{|y|}$$

$$= \frac{-1}{n\alpha(n)}\frac{y}{|y|^n}$$

So for all n:

$$D\Phi(y) = \frac{-1}{n\alpha(n)}\frac{y}{|y|^n}$$

On $\partial B(0,\varepsilon)$: $n = -\frac{y}{\varepsilon}$, $D\Phi(y) = \frac{-1}{n\alpha(n)}\frac{y}{\varepsilon^n}$

so that:

$$n \cdot D\Phi(y) = \frac{|y|^2}{n\alpha(n)}\frac{1}{\varepsilon^{n+1}} = \frac{1}{n\alpha(n)\varepsilon^{n-1}}$$

$$= \frac{1}{\text{area of}(\partial B(0,\varepsilon))} \; !$$

We have that:

$$K_\varepsilon = -\int_{\partial B(0,\varepsilon)} D\Phi(y)\cdot n \, f(x-y)\, dS(y)$$

$$= -\int_{\partial B(0,\varepsilon)} \frac{1}{n\alpha(n)\varepsilon^{n-1}} f(x-y)\, dy$$

$$= \frac{-1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y)\,dy$$ ← This is the average of f in the spherical shell $\partial B(0,\varepsilon)$

$$=: \fint_{\partial B(0,\varepsilon)} f(x-y)\,dy \longrightarrow f(x) \text{ as } \varepsilon \to 0$$

since f is continuous.

Let $\varepsilon \to 0$ to conclude that $\Delta u(x) = -f(x)$ ∎

MEAN VALUE FORMULAS

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u harmonic, $\Delta u = 0$ (in U)

Define $$\varphi(r) = \fint_{\partial B(x,r)} u(y)\,dS(y)$$

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u(y)\,dS(y)$$ $y = x + rz$, $dS(y) = r^{n-1}\,dS(z)$, $z \in \partial B(0,1)$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz)\,dS(z).$$

Then

$$\varphi'(r) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} Du(x+rz)\cdot z\,dS(z)$$ change back to y

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} Du(y)\cdot \mathbf{n}\,dS(y)$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy = 0.$$

Therefore: $\varphi(r) = \lim_{r \to 0} \varphi(r) = \lim_{r \to 0} \int_{\partial B(x,r)} u(y)\, dS(y)$

$= u(x)$ Since u is continuous.

Conclude:

> u harmonic in U, $B(x,r) \subset U$
> $\Rightarrow u(x) = \fint_{\partial B(x,r)} u(x)\, dS(y)$

MEAN VALUE FORMULA

Assume now that u satisfies the mean value property, and that u is not harmonic.

Then there must be a ball $B(x,r)$ on which $\Delta u(y) > 0$.

But

$$0 = \varphi'(r) = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy > 0. \quad !$$

$0 = \varphi'(r)$: mean value property

> 0: contradiction.

> $u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y) \Rightarrow u$ harmonic

MEAN VALUE FORMULA

$$\int_{B(x,r)} u(y)\, dy = \int_0^r \left(\int_{\partial B(x,s)} u(y)\, dS(y) \right) ds = u(x) \int_0^r n\alpha(n) s^{n-1}\, ds$$

$$= u(x)\, \alpha(n) r^n.$$

so that:

$$\Delta u = 0 \iff u(x) = \fint_{B(x,r)} u(y)\,dy$$

MEAN VALUE FORMULA

Maximum principle

Theorem. u harmonic in U, $u \in C(\bar{U}) \cap C^2(U)$.

a) $\max_{\bar{U}} u = \max_{\partial U} u$

b) if $u(x_0) = \max_{\bar{U}} u$ and $x_0 \in U$, then $u(x) = u(x_0)$ for all $x \in \bar{U}$.

Proof. $M = \max_{\bar{U}} u$. Assume $u(x_0) = M$ for some $x_0 \in U$. Let $r < \operatorname{dist}(x_0, \partial U)$.

$$M = u(x_0) = \fint_{B(x_0,r)} u\,dx \leq M \Rightarrow u(x) = M \text{ for all } x \in B(x_0, r).$$

Thus the set $\{x \in U \mid u(x) = M\}$ is open in U, since u is continuous it is also relatively closed. Since U is connected it is either empty or equal to U.

This proves b) $\Rightarrow$ a).

∎

Estimating the derivatives

Let η^ε be the standard mollifier.

$$u_\varepsilon(x) = \int_{B(x,\varepsilon)} \eta^\varepsilon(x-y)\, u(y)\, dy \qquad u_\varepsilon \text{ is } C^\infty(U^\varepsilon)$$

$$= \int_{B(x,\varepsilon)} \frac{1}{\varepsilon^n} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\, dy$$

polar coordinates

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \left(\int_{\partial B(x,r)} u(y)\, dS(y) \right) dr$$

mean value property

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) u(x)\, n\alpha(n)\, r^{n-1}\, dr$$

$$= \frac{u(x)}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \int_{\partial B(x,r)} dS(y)\, dr$$

polar to Cartesian coordinates

$$= u(x) \int_{B(x,\varepsilon)} \eta^\varepsilon(x)\, dx = u(x) \quad !$$

$$\Rightarrow \boxed{u \in C^\infty(U)}$$

We bound the derivatives of u.

α multi-index $|\alpha| = k \geq 0$

u harmonic in U, $x_0 \in U$, $B(x_0, r) \subset U$.

A) $$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

B) $$C_0 = \frac{1}{\alpha(n)} \qquad C_k = \frac{(2^{n+1} n k)^k}{\alpha(n)}, \quad k = 1, 2, 3, \ldots$$

Proof:

$k = 0$. $$|u(x_0)| = \left| \fint_{B(x_0,r)} u \, dx \right| \leq \frac{1}{\alpha(n)} \frac{1}{r^n} \int_{B(x_0,r)} |u| \, dx = \frac{1}{\alpha(n)} \frac{1}{r^n} \|u\|_{L^1(B(x_0,r))}$$

$k = 1$

$D^\alpha u = u_{x_i}$, observe that u_{x_i} is harmonic.

$$|u_{x_i}(x_0)| = \left| \frac{2^n}{\alpha(n) r^n} \int_{B(x_0, r/2)} u_{x_i} \, dx \right| = \frac{2^n}{\alpha(n) r^n} \left| \int_{\partial B(x_0, r/2)} u n_i \, dS \right|$$

↑ integration by parts: reduce the number of derivatives

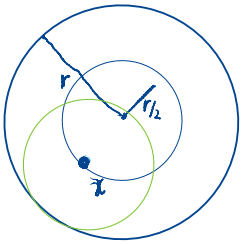
$$\leq \frac{2^n}{\alpha(n) r^n} n \alpha(n) \frac{r^{n-1}}{2^{n-1}} \|u\|_{L^\infty(\partial B(x_0, r/2))}$$

$$= \frac{2n}{r} \|u\|_{L^\infty(\partial B(x_0, r/2))} \quad (*)$$

$x \in \partial B(x_0, r/2)$

$\Rightarrow B(x, r/2) \subset B(x_0, r) \subset U$.

use estimate for $k = 0$ on $B(x, r/2)$



$$|u(x)| \le \frac{1}{\alpha(n)} \frac{2^n}{r^n} \|u\|_{L^1(B(x,r/2))}$$

$$\Rightarrow \|u\|_{L^\infty(\partial B(x_0,r/2))} \le \frac{2^n}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,r))}$$

Insert this in (★)

$$|u_{x_i}(x_0)| \le \frac{2n}{r} \frac{2^n}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,r))}$$

$$= \frac{2^{n+1} n}{\alpha(n)} \frac{1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))}$$

this shows the estimate for $|\alpha| = 1$.

Induction step

Assume bound holds for k. $|\alpha| = k+1$, $\alpha = \beta + e_i$.

$D^\alpha u$ is harmonic.

$$|D^\alpha u(x_0)| = |D^\beta u_{x_i}(x_0)| = \left| \frac{(k+1)^n}{\alpha(n) r^n} \int_{B(x_0, r/k+1)} D^\beta u_{x_i} \, dx \right|$$

$$\le \frac{(k+1)^n}{\alpha(n) r^n} \int_{\partial B(x_0, r/k+1)} |D^\beta u \, n_i| \, dS$$

$$\le \frac{(k+1)^n}{\alpha(n) r^n} n \alpha(n) \frac{r^{n-1}}{(k+1)^{n-1}} \|D^\beta u\|_{L^\infty(\partial B(x_0, r/k+1))}$$

(★)

$$= \frac{(k+1) n}{r} \|D^\beta u\|_{L^\infty(\partial B(x_0, r/k+1))}. \quad (★)$$

$x \in \partial B(x_0, r/k+1)$

$$\Rightarrow B\left(x, \frac{k}{k+1} r\right) \subset B(x_0, r) \subset U.$$

induction hypothesis

$$|D^\beta u(x)| \le \frac{(2^{n+1} n k)^k}{\alpha(n)} \frac{(k+1)^{n+k}}{k^{n+k}} \frac{1}{r^{n+k}} \|u\|_{L^1(B(x, \frac{k}{k+1} r))}$$

By (★)

$$|D^\alpha u(x_0)| \le \frac{(k+1)n}{r} \frac{(2^{n+1})^k n^k k^k (k+1)^{n+k}}{\alpha(n) \; k^{n+k} \; r^{n+k}} \|u\|_{L^1(B(x_0, r))}$$

$$= \frac{(2^{n+1})^k n^{k+1} (k+1)^{k+1}}{\alpha(n) r^{n+k+1}} \left(\frac{k+1}{k}\right)^n \|u\|_{L^1(B(x_0,r))}$$

$$\left(\frac{k+1}{k}\right)^n \le 2^{n+1}$$

$$\le \frac{(2^{n+1} n(k+1))^{k+1}}{\alpha(n)} \frac{1}{r^{n+k+1}} \|u\|_{L^1(B(x_0,r))}$$

Liouville's theorem

Suppose $u: \mathbb{R}^n \to \mathbb{R}$ is bounded and harmonic. Then u is constant.

Proof:

$|Du|^2 = \sum_{i=1}^n u_{x_i}^2$, use previous bound on u_{x_i}.

to get

$$|Du(x_0)| \le \frac{\sqrt{n} C_1}{r^{n+1}} \int_{B(x_0,r)} |u| \, dx \le \frac{\sqrt{n} C_1 r^n \alpha(n)}{r^{n+1}} \|u\|_{L^\infty(\mathbb{R}^n)} \to 0 \text{ as } r \to \infty$$

$\Rightarrow Du = 0 \Rightarrow u$ is a constant. ∎

Harmonic in U $\Rightarrow$ analytic in U

$r = \frac{1}{4}\operatorname{dist}(x_0, \partial U)$ $\qquad M := \frac{1}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,r))}$

$x \in B(x_0, r) \Rightarrow B(x, r) \subset U$

$$|D^\alpha u(x)| \leq \frac{(2^{n+1} n|\alpha|)^{|\alpha|}}{\alpha(n)} \frac{1}{r^{|\alpha|+n}} \|u\|_{L^1(B(x,r))}$$

$$\leq M \left(\frac{2^{n+1} n}{r}\right)^{|\alpha|} |\alpha|^{|\alpha|}$$

this gives an L^∞ bound since RHS indep. on x.

Facts: $|\alpha|^{|\alpha|} \leq e^{|\alpha|} |\alpha|!$

$$n^k = (1 + \ldots + 1)^k = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} \Rightarrow |\alpha|! \leq n^{|\alpha|} \alpha! \; ; |\alpha| = k$$

$$\Rightarrow |\alpha|^{|\alpha|} \leq n^{|\alpha|} \alpha! \, e^{|\alpha|}$$

so $$\boxed{\|D^\alpha u\|_{L^\infty(B(x_0,r))} \leq M \left(\frac{2^{n+1} n^2 e}{r}\right)^{|\alpha|} \alpha!}$$

Taylor series with remainder

$$u(x) = \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha u(x_0) (x - x_0)^\alpha$$

$$+ \sum_{|\alpha|=N} \frac{1}{\alpha!} D^\alpha u(x_0 + t(x - x_0)) (x - x_0)^\alpha, \quad t \in (0,1).$$

↑ Remainder term. R_N

must show that $R_N \to 0$ as $N \to \infty$

$$|R_N(x)| \le \sum_{|\alpha|=N} M\left(\frac{2^{n+1} n^2 \ell}{r}\right)^{|\alpha|} |(x-x_0)^\alpha|$$

choose $|x-x_0| \le \frac{r}{2^{n+2} n^3 \ell}$

$$\le \sum_{|\alpha|=N} M\left(\frac{2^{n+1} n^2 \ell}{r}\right)^{|\alpha|} \left(\frac{r}{2^{n+2} n^3 \ell}\right)^{|\alpha|}$$

$$\le \sum_{|\alpha|=N} M\left(\frac{2^{n+1} n^2 \ell}{r}\right)^{|\alpha|} \left(\frac{r}{2^{n+2} n^3 \ell}\right)^{|\alpha|}$$

$$\le M\, n^N \frac{1}{(2n)^N} \le M \frac{1}{2^N} \to 0$$

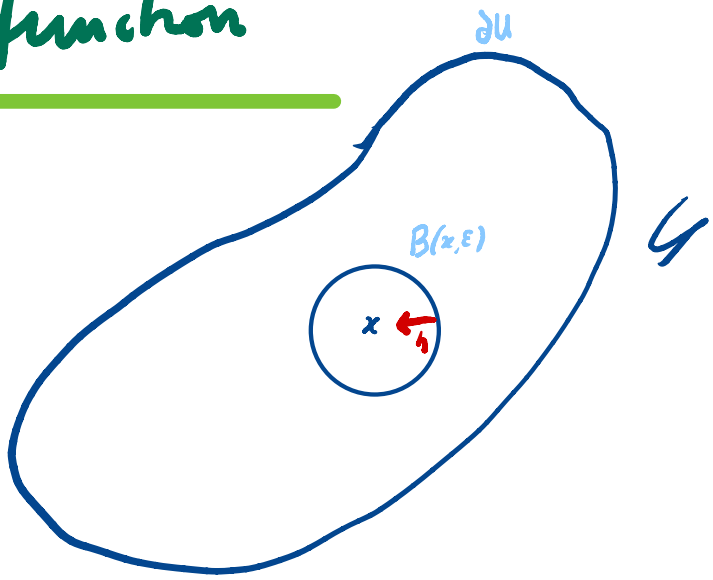
∎

Nth order derivatives of u

$\le n^N$

each of the N derivatives can be in n different places.

Green's function



$u \in C^2(\bar{U})$

$V_\varepsilon = U \setminus B_0(x,\varepsilon)$

Φ : fundamental solution.

integration by parts.

$$\text{LHS}: \int_{V_\varepsilon} u(y)\,\Delta\Phi(y-x) - \Delta u(y)\,\Phi(y-x)\,dx$$

$$= \int_{\partial V_\varepsilon} u(y)\, D\Phi(y-x)\cdot n - Du(y)\cdot n\, \Phi(y-x)\, dS(y) \quad (*)$$

$$\doteq \text{RHS}$$

$$\partial V_\varepsilon = \partial U \cup \partial B(x,\varepsilon).$$

$$\Delta\Phi(y-x) = 0 \quad \text{for} \quad y \in V_\varepsilon.$$

$$\text{LHS} = -\int_{V_\varepsilon} \Delta u(y)\,\Phi(x-y)\,dy \longrightarrow -\int_U \Delta u(y)\,\Phi(y-x)\,dy \quad \varepsilon \to 0.$$

$$\text{RHS} \quad \int_{\partial B(x,\varepsilon)} u(y)\, D\Phi(y-x)\cdot n\, dS(y) = \fint_{\partial B(x,\varepsilon)} u(y)\, dS(y) \longrightarrow u(x) \quad \varepsilon \to 0.$$

$$\int_{\partial B(x,\varepsilon)} Du(y)\cdot n\, \Phi(y-x)\, dS(y) \sim C\varepsilon \longrightarrow 0 \quad \varepsilon \to 0.$$

$$\lim_{\varepsilon\to 0} RHS = -u(x) + \int_{\partial U} u(y)\, D\Phi(y-x)\cdot n - Du(y)\cdot n\, \Phi(y-x)\, dS(y)$$

Collecting this:

B1

$$u(x) = \int_{\partial U} \Phi(y-x)\, Du(y)\cdot n - D\Phi(y-x)\cdot n\, u(y)\, dS - \int_U \Phi(y-x)\, \Delta u(y)\, dy$$

if u solves Poisson's equation:

$$\begin{cases} -\Delta u = f & U \\ u = g & \partial U \end{cases}$$

this would almost give a formula for u, except for the troublesome part:

$$Du\cdot n\big|_{\partial U}$$ we do not know this ...

The remedy:

Define a corrector:

For each $x \in U$, define:

$$\begin{cases} \Delta \varphi^x(y) = 0 & y \in U \\ \varphi^x(y) = \Phi(y-x) & y \in \partial U \end{cases}$$

Assume we can solve this.

Use integration by parts with u and φ^x:

$$-\int_U \varphi^x(y)\,\Delta u(y)\,dy = \int_{\partial U} u(y)\,D\varphi^x(y)\cdot n - \varphi^x(y)\,Du(y)\cdot n \; dS(y) \qquad \text{B2}$$

$$= \int_{\partial U} u(y)\,D\varphi^x(y)\cdot n - \Phi(y-x)\,Du(y)\cdot n \; dS(y)$$

bad term!

Add B1 and B2 and tidy

$$u(x) = -\int_U (\Phi(y-x) - \varphi^x(y))\,\Delta u(y)\,dy$$

$$-\int_{\partial U} D(\Phi(y-x) - \varphi^x(y))\cdot n\; u(y)\; dS(y)$$

Define the Greene's function

$$G(x,y) = \Phi(y-x) - \varphi^x(y)$$

With this notation, the solution of Poisson's equation is

$$u(x) = \int_U f(y)\, G(x,y)\, dy - \int_{\partial U} g(y)\, D_y G(x,y) \cdot \mathbf{n}\, dS(y)$$

G is symmetric, i.e., $G(x,y) = G(y,x)$ Proof in Evans!

The Dirichlet problem

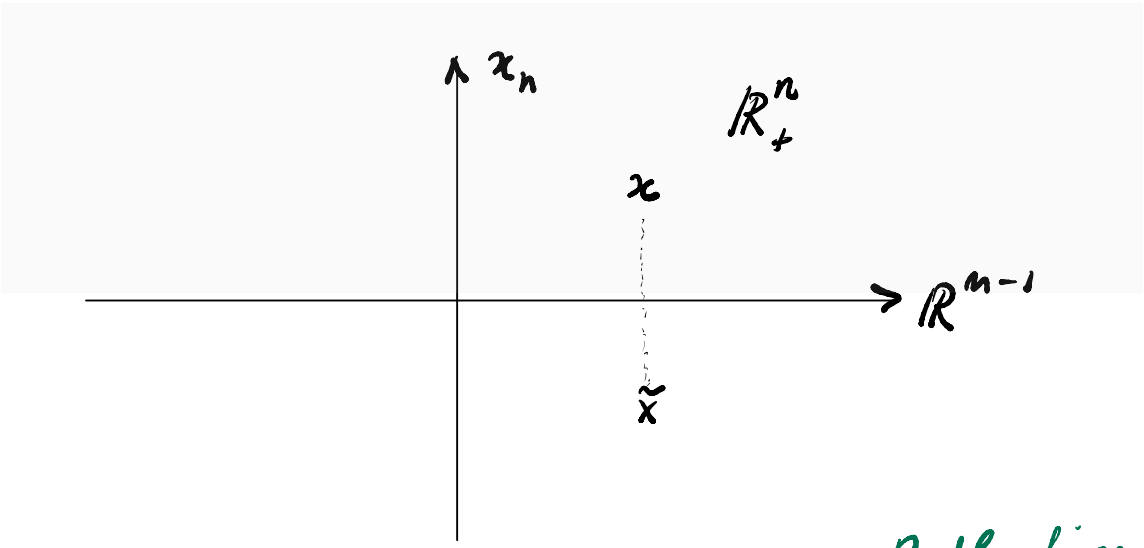
$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

formula with Green's function:

$$u(x) = -\int_{\partial U} g(y)\, D_y G(x,y) \cdot \mathbf{n}\, dS(y)$$

Examples

$$U = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$$



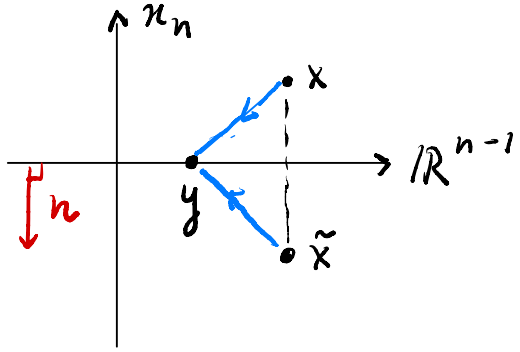
$$\tilde{x} = (x_1, x_2, \dots, x_{n-1}, -x_n) \quad \leftarrow \text{Reflection}$$

Define

$$\varphi^x(y) = \Phi(y - \tilde{x})$$

since $\tilde{x} \notin \mathbb{R}^n_+$, $\varphi^x(y)$ harmonic for $y \in \mathbb{R}^n_+$.

$\varphi^x(y)$ for $y \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$?



$$\Rightarrow \Phi(y - \tilde{x}) = \Phi(y - x) \; !$$

To get a solution formula, we must find

$$D_y G(x, y) \cdot n$$

$$n = -e_n$$

$$G(x,y) = \Phi(y-x) - \Phi(y,\tilde{x})$$

$$D_y G(x,y) = D_y\Phi(y-x) - D_y\Phi(y-\tilde{x}) \quad | \cdot n$$

$$D_y G(x,y)\cdot n = -\frac{\partial}{\partial y_n}\left[\Phi(y-x) - \Phi(y-\tilde{x})\right] \quad \text{for } y_n = 0$$

$$D_y\Phi = \frac{-1}{n\alpha(n)}\frac{y-x}{|y-x|^n}$$

$$= \frac{-1}{n\alpha(n)}\left[\frac{y_n - x_n}{|x-y|^n} - \frac{y_n + x_n}{|\tilde{x}-y|^n}\right]$$

$y_n = 0$

$|y-x| = |y-\tilde{x}|$

$$= \frac{2x_n}{n\alpha(n)|y-x|^n} \qquad y \in \mathbb{R}^{n-1}$$

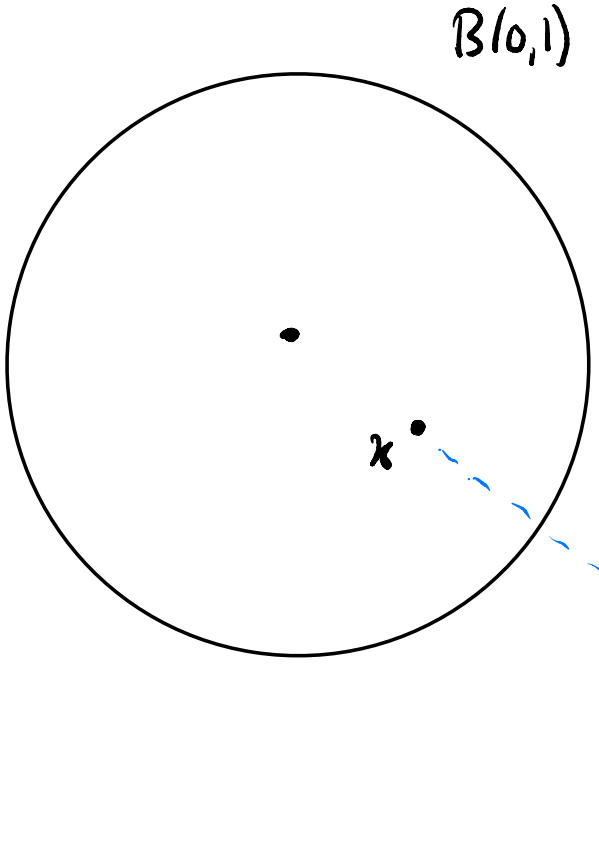
$$u(x) = \frac{2x_n}{n\alpha(n)}\int_{\mathbb{R}^{n-1}} \frac{g(y)}{|y-x|^n}\,dy$$

Singular term!!

Must check formula!



Green's function for a ball



$$\tilde{x} = \frac{x}{|x|^2}$$

Note: $x \in \partial B(0,1) \Rightarrow x = \tilde{x}$

$$\begin{cases} \Delta u = 0 & B(0,1) \\ u = g & \partial B(0,1) \end{cases}$$

$v(x) = u\left(\frac{x}{r}\right)$ $x \in B(0,r)$, solves:

$$\begin{cases} \Delta v = 0 & B(0,r) \\ v = g\left(\frac{x}{r}\right) & \partial B(0,r) \end{cases}$$

Define $\varphi^x(y) = \Phi(|x|(y-\tilde{x}))$

$y \longmapsto \varphi^x(y)$ harmonic for $y \in B(0,1)$

$|y|=1 \leftrightarrow y \in \partial B(0,1)$

$$||x|(y-\tilde{x})|^2 = |x|^2\left(y-\frac{x}{|x|^2}\right)\cdot\left(y-\frac{x}{|x|^2}\right)$$

$$= |x|^2\left(y\cdot y - \frac{2x\cdot y}{|x|^2} + \frac{x\cdot x}{|x|^4}\right)$$

$$= |x|^2 - 2x\cdot y + 1$$

$$= x\cdot x - 2x\cdot y + y\cdot y = |x-y|^2$$

so for $|y|=1$

$$\Phi(|x|(y-x)) = \Phi(y-x)\ !$$

To get a formula we compute:

$n = y$ for $|y|=1$.

$$G_{y_i}(x,y) = \Phi_{y_i}(y-x) - \Phi(|x|(y-\tilde{x}))_{y_i}$$

$$= \frac{1}{n\alpha(n)}\frac{x_i-y_i}{|x-y|^n} + \frac{1}{n\alpha(n)}\frac{1}{||x|(y-\tilde{x})|^n}|x|^2\left(y_i - \frac{x_i}{|x|^2}\right)$$

$|y|=1$

$$= \frac{1}{n\alpha(n)|y-x|^n}\left[(x_i-y_i) + y_i|x|^2 - x_i\right]$$

$$= \frac{1}{n\alpha(n)|y-x|^n}\, y_i(|x|^2-1)$$

we get for $|y|=1$

$$D_y G(x,y)\cdot n = \sum_{i=1}^{n} y_i\, G_{y_i}(x,y)$$

$$= \frac{1}{n\alpha(n)|y-x|^n}(|x|^2-1)\, y_i y_i$$

$$= \frac{-1}{n\alpha(n)}(1-|x|^2)\frac{1}{|y-x|^n}$$

Formula:

$$u(x) = \frac{1-|x|^2}{n\alpha(n)}\int_{\partial B(0,1)} \frac{g(y)}{|y-x|^n}\, dS(y)$$

Poisson's formula for a ball

singular term! must check



Energy methods

$$\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

Thm.

This problem has at most one solution.

Proof: Let u_1 and u_2 be two solutions

set $w = u_1 - u_2$, then

$$\begin{cases} \Delta w = 0 & \Omega \\ w = 0 & \partial\Omega \end{cases}$$

$$0 = -\int_\Omega w \Delta w \, dy = \int_\Omega |Dw|^2 \, dy \qquad \text{IBP}$$

$$Dw = 0 \Rightarrow w = \text{const.}$$

$$w|_{\partial\Omega} = 0 \Rightarrow w = 0$$

∎

Variational formulation

Energy functional

$$I[u] = \int_U \frac{1}{2}|Du|^2 - fu \, dy$$

domain of I:

$$\mathcal{A} = \{ u \in C^2(\bar{U}) \mid u = g \text{ on } \partial U \}.$$

We want to find a function u which minimizes I;

$$u = \arg\min_{\mathcal{A}} I[v]$$

Set $f(\tau) = I[u + \tau v]$.

If u is a local minimum, then

$$f'(\tau)\Big|_{\tau=0} = 0.$$

let $v \in C^2(\bar{U})$ s.t. $v\big|_{\partial U} = 0$, $u + \tau v \in \mathcal{A}$

$$f(\tau) = I[u + \tau v]$$

$$= \int_U \frac{1}{2}|Du + \tau Dv|^2 - f\cdot(u + \tau v) \, dy$$

$$= \int_U \tfrac{1}{2}|Du|^2 + \tau\, Du \cdot Dv + \tau^2 |Dv|^2 - fu - \tau f v \, dy$$

$$0 = f'(\tau)\big|_{\tau=0} = \int_U Du \cdot Dv - fv \, dy$$

IBP!

$$= -\int_U (\Delta u + f) v \, dy$$

since this is to hold for all v

we get

$$-\Delta u = f \quad \text{in } U.$$