Notation and useful facts

$x \in \mathbb{R}^n \qquad x = (x_1, x_2, \dots x_n)$

$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$ Euclidian norm

Functions

$u : \mathbb{R}^n \to \mathbb{R} \qquad u(x) = u(x_1, \dots, x_n)$

Lipschitz continuity

$|u(x) - u(y)| \le C |x - y| \qquad C < \infty$

$|u|_{Lip} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|}$

Differentiation

$$\frac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \frac{u(x + h e_i) - u(x)}{h}$$

$u_{x_i} := \frac{\partial u}{\partial x_i} \qquad u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ etc...

Multi-indeces

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \qquad \alpha_i \in \{0, 1, 2, \dots\}$

$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

$k \in \mathbb{N}$

$$D^k u = \{ D^\alpha u \mid |\alpha| = k \}$$ all derivatives of order k

$$Du = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$$ ← The gradient of u

$$D^2 u = \begin{pmatrix} u_{x_1x_1} & u_{x_1x_2} & \cdots & u_{x_1x_n} \\ u_{x_2x_1} & u_{x_2x_2} & & \\ \vdots & & \ddots & \\ u_{x_nx_1} & & & u_{x_nx_n} \end{pmatrix}$$ ← The Hessian of u

The ball or radius r

$$B(x,r) = \{ y \in \mathbb{R}^n \mid |x-y| \leq r \}$$ closed!

$$B^\circ(x,r) = B(x,r) \setminus \partial B(x,r), \quad r > 0.$$

volume of $B(x,r) = \alpha(n) r^n$

$\alpha(n)$ volume of the unit sphere in $\mathbb{R}^n$

fun fact →

$$\alpha(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ \frac{2\pi}{d}\alpha(n-2) & \text{otherwise} \end{cases}$$

$\alpha(n) \to 0$ as $n \to \infty$

$n\alpha(n) r^{n-1}$ ← surface "area" of $\partial B(x,r)$

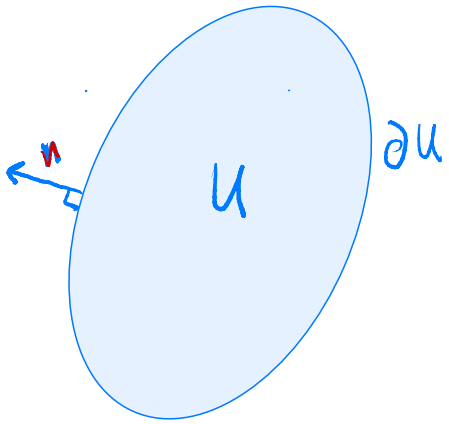
$n\alpha(n)$ Surface area of the unit sphere

Integration by parts

$U \subset\subset \mathbb{R}^n$, smooth boundary ∂U

n unit normal

$|n| = 1$



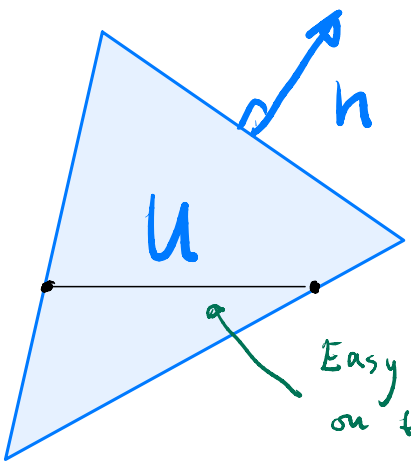
$u : U \to \mathbb{R}$

$$\int_U u_{x_i}\, dx = \int_{\partial U} u n_i\, dS$$

Surface integral

"Proof"

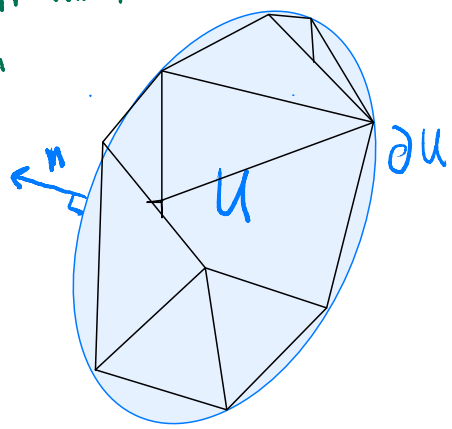
Easy to prove if U is a "triangle"



Then "sum" all the black lines...

Easy 1D integration by parts on the black line

For general U, approximate U by a collection of triangles



The integrals along the inner lines cancel.

Finally, use very small triangles. ∎

Consequences:

$u = (u^1, u^2, \dots u^n)$ a vector field on $\mathbb{R}^n$.

$$\operatorname{div}(u) := u^1_{x_1} + u^2_{x_2} + \dots u^n_{x_n}$$

$$\int_U \operatorname{div}(u)\,dx = \int_U \sum_{i=1}^n u^i_{x_i}\,dx = \int_{\partial U} \sum_{i=1}^n u^i n_i\,dS$$

$$= \int_{\partial U} u \cdot n\,dS$$

Observe that

$$\operatorname{div}(\operatorname{grad} u) = \operatorname{div}(Du) = \sum_{i=1}^n u_{x_i x_i} = \operatorname{trace}(D^2 u)$$

$$= \Delta u$$

$$\boxed{\int_U \Delta u\,dx = \int_{\partial U} Du \cdot n\,dS}$$

Leibnitz rule: $(uv)_{x_i} = uv_{x_i} + u_{x_i} v$

$$\int_U (uv)_{x_i} \, dx = \int_{\partial U} (uv) n_i \, dS$$

Use this with $v = w_{x_i}$

$$\int_U (u w_{x_i})_{x_i} \, dx = \int_{\partial U} u \, w_{x_i} n_i$$

Leibnitz rule on the left

$$\int_U u w_{x_i x_i} + u_{x_i} w_{x_i} \, dx = \int_{\partial U} u \, w_{x_i} \, dS$$

Sum over i:

$$\int_U u \Delta w + Du \cdot Dw \, dx = \int_{\partial U} u \, Dw \cdot n \, dS$$

By symmetry

$$\int w \Delta u + Du \cdot Dw \, dx = \int_{\partial U} w \, Du \cdot n \, dS$$

Subtract

$$\boxed{\int_U v \Delta u - u \Delta v \, dx = \int_{\partial U} v \, Du \cdot n - u \, Dv \cdot n \, dS}$$

$v \equiv 1$

$$\boxed{\int_U \Delta u \, dx = \int_{\partial U} Du \cdot n \, dS}$$

Polar coordinates

$$\int_{\mathbb{R}^n} f\,dx = \int_0^\infty \left(\int_{\partial B(0,r)} f\,dS \right) dr$$

$$\frac{d}{dr}\left(\int_{B(0,r)} f\,dx \right) = \int_{\partial B(0,r)} f\,dS$$

Mollifiers

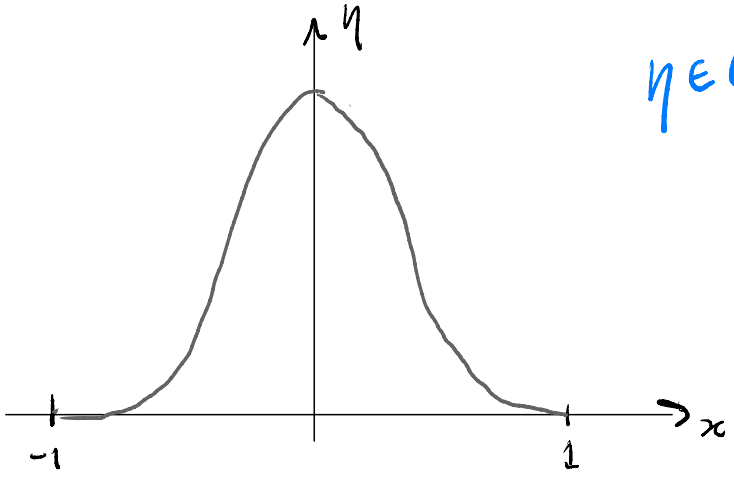
$$\eta(x) = \begin{cases} C\, e^{\frac{1}{|x|^2-1}} & |x|<1 \\ 0 & |x|\geq 1 \end{cases}$$

Prove this!

$\eta \in C^\infty(\mathbb{R})$

Choose C s.t.

$$\int_{\mathbb{R}} \eta(x)\,dx = 1$$



η is called a standard mollifier

Define $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{|x|}{\varepsilon}\right)$, $\varepsilon > 0$.

$U \subseteq \mathbb{R}^n$ $f: U \to \mathbb{R}$ f locally integrable.

$U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}$.

Define

$$f^\varepsilon(x) = \int_U \eta_\varepsilon(x-y) f(y)\, dy$$

$$f^\varepsilon : U_\varepsilon \to \mathbb{R}$$

Note:

$$f^\varepsilon(x) = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) f(x-y)\, dy$$

Prove this with a change of variables!

Facts of mollification

1) $f^\varepsilon \in C^\infty(U_\varepsilon)$
2) $f^\varepsilon \to f$ a.e. x as $\varepsilon \to 0$.
3) If $f \in C(U)$, $f^\varepsilon \to f$ uniformly on compacts
4) Convergence also in $L^p_{loc}(U)$

Proved in Evans