MAT 4301 Partial differential eqns.

contents: Poisson equation
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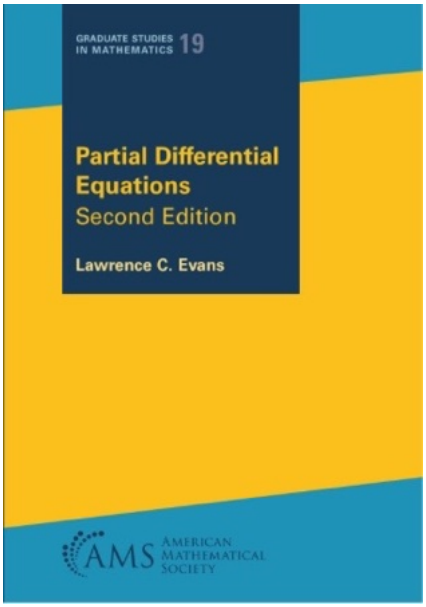
What's new?

Poisson equation, Heat equation } in $\mathbb{R}^n$, not only in $\mathbb{R}$

Conservation laws, Hamilton-Jacobi } non-linear

Material:

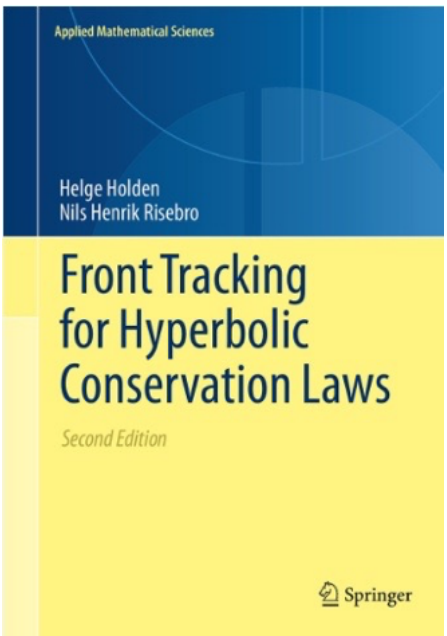
Partial differential equations



Chapter 2, ++

- expensive, but very good...
- also used in other courses

Front tracking for hyperbolic conservation laws



Chapters 1, 2, ++

- available online at UiO
- good if you are planing to write MSc thesis at UiO.

The situation in 1D

Harmonic functions; $u''(x) = 0$.

$$\begin{cases} -u''(x) = 0 & x \in (0,1) \\ u(0) = a\,, \; u(1) = b \end{cases}$$

Solution: $u(x) = a + x(b-a)$

Maximum principle:

$$\min(a,b) \le u(x) \le \max(a,b)$$

We shall generalize this to $\mathbb{R}^n$...

The Poisson equation:

$$\begin{cases} -u''(x) = f(x) & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

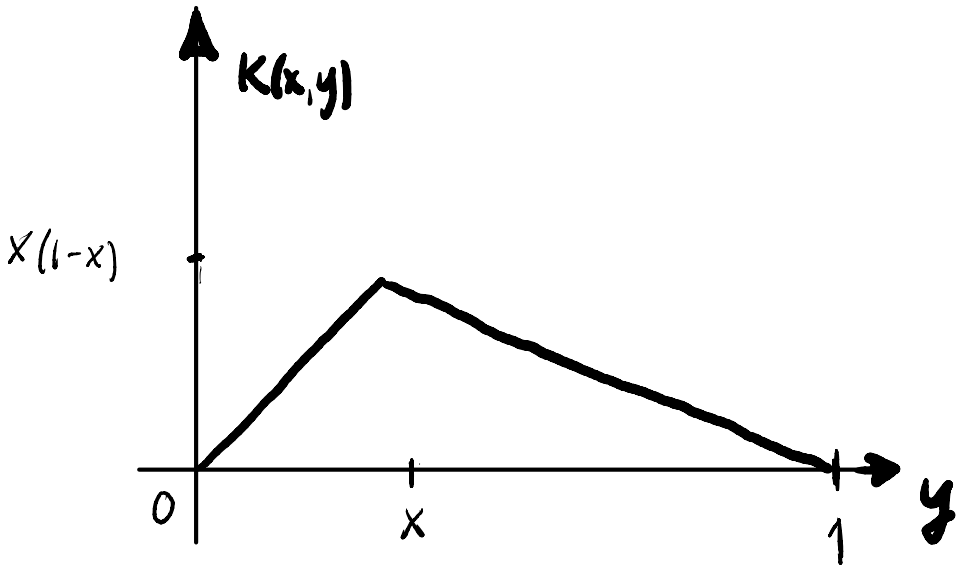
f is a given function.

Solving the Poisson equation in 1D.

Define

$$K(x,y) = \begin{cases} y(1-x) & \text{if } 0 \le y \le x \\ (1-y)x & \text{if } x \le y \le 1 \end{cases}$$

$(x,y) \in [0,1] \times [0,1]$.



Observe that

$$K(0,y) = K(1,y) = 0.$$

Now we claim that

$$u(x) = \int_0^1 K(x,y) f(y)\, dy$$

solves the Poisson equation.

Proof: We first check the boundary conditions:

$$u(0) = \int_0^1 K(0,y) f(y)\,dy = 0, \quad u(1) = \int_0^1 K(1,y) f(y)\,dy = 0$$ ok!

We also have:

$$\begin{aligned} u(x) &= \int_0^1 K(x,y) f(y)\,dy \\ &= \int_0^x K(x,y) f(y)\,dy + \int_x^1 K(x,y) f(y)\,dy \\ &= \int_0^x y(1-x) f(y)\,dy + \int_x^1 x(1-y) f(y)\,dy. \end{aligned}$$

Therefore:

$$\begin{aligned} u'(x) &= x(1-x) f(x) + \int_0^x -y f(y)\,dy \\ &\quad - x(1-x) f(y) + \int_x^1 (1-y) f(y)\,dy \\ &= -\int_0^x y f(y)\,dy + \int_x^1 (1-y) f(y)\,dy, \end{aligned}$$

which implies

$$u''(x) = -x f(x) - (1-x) f(x) = -f(x).$$

■

K is called the Green function (for this problem)

This leads to other bounds:

$$|u(x)| \le \int_0^1 K(x,y)\,|f(y)|\,dy$$

$$\le \max_y |f(y)| \int_0^1 K(x,y)\,dy \qquad \int_0^1 K(x,y)\,dy = \tfrac{1}{2}x(1-x)$$

$$= \|f\|_\infty \tfrac{1}{2} x(1-x)$$

$$\le \tfrac{1}{8}\|f\|_\infty .$$

Uniqueness?

Let v be another solution s.t.

$$\begin{cases} -v'' = f & x \in (0,1) \\ v(0) = v(1) = 0 \end{cases}$$

Linearity!

Set $w = u - v$ $\longrightarrow$ $\begin{cases} -w'' = 0 & (0,1) \\ w(0) = w(1) = 0 \end{cases}$

$$0 = \int_0^1 w w''\,dx = w(x)w'(x)\Big|_0^1 - \int_0^1 |w'(x)|^2\,dx$$

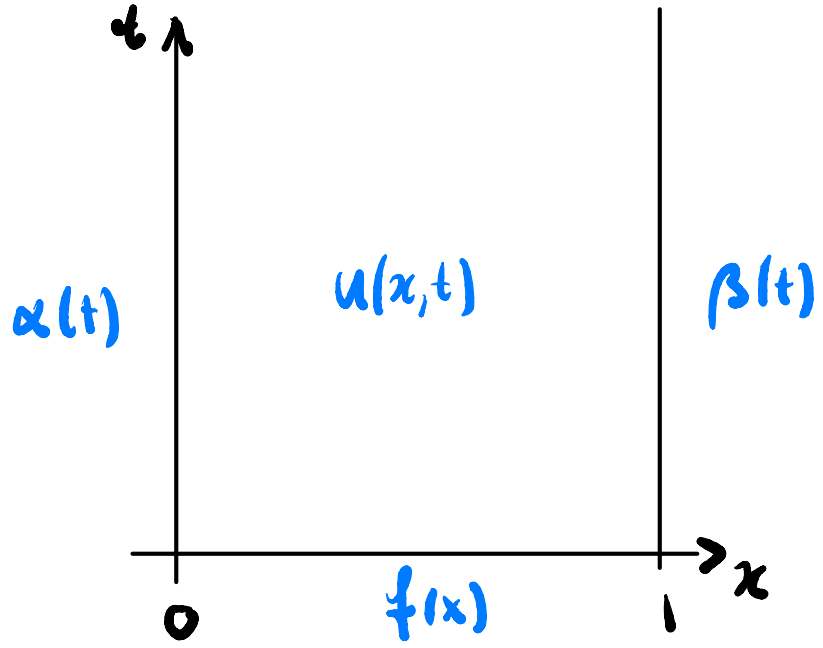
$$\Rightarrow \int_0^1 |w'(x)|^2\,dx = 0$$

$$\Rightarrow w'(x) = 0 \Rightarrow w(x) = C$$

but $0 = w(0) = C$, so $w(x) = 0$, and $u = v$!

The heat equation on the unit interval

$$(H)\quad \begin{cases} u_t(x,t) = u_{xx}(x,t) & t>0 \quad x\in(0,1) \\ u(0,t)=\alpha(t) \quad u(1,t)=\beta(t) \\ u(x,0)=f(x) \end{cases}$$



$U_T = (0,1)\times(0,T]$ parabolic cylinder

$\partial U_T = \overline{U_T} \setminus U_T$

"Data" on ∂U_T

Solution on U_T

Maximum principle: If u solves H, then:

$$u(x,t) \leq \max_{(x,t)\in\partial U_t} \{\alpha(t), \beta(t), f(x)\}$$

Proof:

Assume that u has a max in $(\hat{x},\hat{t}) \in U_t$ and try to get a contradiction:

Since u has a max at $(\hat{x},\hat{t})$

$$\frac{\partial u}{\partial t}(\hat{x},\hat{t}) \geq 0 \qquad \frac{\partial^2 u}{\partial x^2}(\hat{x},\hat{t}) \leq 0$$

$$\Rightarrow \quad 0 \leq u_t(\hat{x},\hat{t}) = u_{xx}(\hat{x},\hat{t}) \leq 0 \qquad \text{No contradiction :(}$$

Introduce auxilliary function

$$v(x,t) = u(x,t) + \varepsilon t, \qquad \varepsilon > 0$$

$$\Rightarrow \quad v_t = u_t + \varepsilon \qquad v_{xx} = u_{xx}$$

So v solves

$$\begin{cases} v_t = v_{xx} - \varepsilon & U_T \\ v(0,t) = \alpha(t) + \varepsilon t \qquad v(1,t) = \beta(t) + \varepsilon t \\ v(x,0) = f(x) \end{cases}$$

Assume now that v has a max in $(\hat{x},\hat{t}) \in U_t$.

Then:

$$0 \le v_t(\hat{x},\hat{t}) = v_{xx}(\hat{x},\hat{t}) - \varepsilon \le -\varepsilon < 0.$$

Contradiction! Thus

$$v(x,t) \le \max_{\partial U_T} \{\alpha(t) + \varepsilon t, \beta(t) + \varepsilon t, f(x)\}$$

$$\Downarrow$$

$$u(x,t) < v(x,t) \le \max_{\partial U_T} \{\cdots\}$$

Let now $\varepsilon \searrow 0$. ∎

We also have a minimum principle:

$$u(x,t) \ge \min_{(x,t)\in\partial U_t} \{\alpha(t), \beta(t), f(x)\}$$

Prove it!

and

$$|u(x,t)| \le \max\{|\alpha|, |\beta|\}$$

This gives uniqueness:

Let v be a solution with data $\hat{\alpha}, \hat{\beta}, \hat{f}$

the $w := u - v$ satisfies

$$\begin{cases} w_t = w_{xx} & U_T \\ w(0,t) = \Delta\alpha & w(1,t) = \Delta\beta(t) \\ w(x,0) = \Delta f \end{cases}$$

$\Delta\alpha = \alpha - \hat{\alpha}$ etc.

$$\Downarrow$$

$$|w(x,t)| \le \max_{\partial U_T} \{|\Delta\alpha|, |\Delta\beta|, |\Delta f|\}$$

Linearity

Conservation laws:

(CL) $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \qquad x \in \mathbb{R}$

$u = u(x,t)$

Hamilton-Jacoby equations:

(HJ) $\frac{\partial u}{\partial t} + g\left(\frac{\partial u}{\partial x}\right) = 0 \qquad x \in \mathbb{R}$

Initial value problem: $u(x,0) = u_0(x)$.

These are **non-linear** equations if f and g are **non-linear**.

The linear case

$f(u) = au + b \longrightarrow u_t + a u_x = 0$ for CL

$g(u) = \alpha u + \beta \longrightarrow u_t + \alpha u_x = -\beta$ for HJ.

$u(x,t) = u_0(x - at)$ solves CL

Characteristics $\begin{cases} x'(t) = a \\ x(0) = x_0 \end{cases} \qquad x(t) = x_0 + at$

Assume that $u(x,t)$ is a solution of linear CL.

$$\frac{d}{dt} u(x(t),t) = u_t(x(t),t) + u_x(x(t),t)\, x'(t) = 0$$

$$\Rightarrow u(x(t),t) = u_0(x_0) \qquad x = x(t) = at + x_0 \qquad x_0 = x - at$$

i.e., $u(x,t) = u_0(x - at)$ ← travelling wave

$u(x,t) = u_0(x-\alpha t) - \beta t$ solves HJ

Characteristics $\begin{cases} x'(t) = \alpha \\ x(0) = x_0 \end{cases}$ $\quad x(t) = x_0 + \alpha t$

Assume that $u(x,t)$ is a solution of linear HJ.

$$\frac{d}{dt} u(x(t),t) = u_t(x(t),t) + u_x(x(t),t)x'(t)$$

$$= u_t(x(t),t) + \alpha u_x(x(t),t) = -\beta$$

$$\Rightarrow \quad u(x(t),t) = u_0(x_0) - \beta t$$

$$u(x,t) = u_0(x-\alpha t) - \beta t$$