The Laplace equation

$$\Delta u(x) = 0, \quad x \in \mathbb{R}^n$$

We try to find "some kind of formula".

Assume that $u(x) = v(r)$

$$r = |x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

$$r_{x_i} = \frac{\partial}{\partial x_i}(x_1^2 + \dots + x_i^2 + \dots + x_n^2)^{1/2} = \frac{x_i}{r}$$

$$r_{x_i x_i} = \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = \frac{r - x_i \frac{x_i}{r}}{r^2} = (r - \frac{x_i^2}{r})/r^2$$

$$u_{x_i x_i} = v(r)_{x_i x_i} = (v'(r) r_{x_i})_{x_i}$$

$$= v''(r) r_{x_i}^2 + v'(r) r_{x_i x_i}$$

$$= v''(r) \frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \sum_{i=1}^{n} v''(r) \frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{1}{r}\frac{x_i^2}{r^2}\right)$$

$$= v''(r) + v'(r)\left(\frac{n-1}{r}\right) = 0.$$

If $r \neq 0$, $v \neq 0$

$$\frac{v''}{v'} = -\frac{n-1}{r} \qquad n \geq 2.$$

$$\Rightarrow \quad \frac{d}{dr}\log(v') = -(n-1)\frac{d}{dr}\log(r) = \log(r^{1-n})$$

$$\Rightarrow \quad v' = C\, r^{1-n}$$

$$v(r) = \begin{cases} C\log(r) & n=2 \\ C\, r^{2-n} & n \geq 3. \end{cases}$$

Define the fundamental solution:

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|) & n=2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

$\alpha(n)$: volume of the sphere

Observe:

$$\lim_{x \to 0} \Phi(x) = \infty$$

$$|D\Phi(x)| \leq C \frac{1}{|x|^{n-1}} \quad \leftarrow \text{ locally integrable}$$

$$|D^2\Phi(x)| \leq C \frac{1}{|x|^{n}} \quad \leftarrow \text{ not locally integrable near } 0$$

Solving the Poisson equation on $\mathbb{R}^n$

(P) $\quad -\Delta u = f \quad \text{in } \mathbb{R}^n$

Assume that:

i) f has compact support

($f(x) \equiv 0$ if $|x| > R$)

ii) $f \in C^2(\mathbb{R}^n)$

(S) $\quad u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$

Theorem: a) $u \in C^2(\mathbb{R}^n)$

b) u solves (P)

Proof of a):

$$u(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\, dy = \int_{\mathbb{R}^n} K(y) f(x-y)\, dy$$

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \underbrace{\frac{1}{h}\left(f(x+he_i-y) - f(x-y)\right)}_{\downarrow h \to 0} dy$$

$\downarrow h \to 0$

$$u_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(y)\, f_{x_i}(x-y)\, dy$$

same method for the second order derivatives. ∎

Proof of b):

$$\Delta u(x) = \int_{\mathbb{R}^n} \Phi(y)\, \Delta_x f(x-y)\, dy$$

$$= \underbrace{\int_{B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy}_{I_\varepsilon} + \underbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy}_{J_\varepsilon}$$

we want to make this small.

we have

$n=2$

$$\int_{B(0,\varepsilon)} |\Phi(y)|\, dy = C' \int_{B(0,\varepsilon)} |\log(r)|\, dy = C \int_0^\varepsilon r \log(r)\, dr \leq C\varepsilon^2 \log(\varepsilon)$$

polar coordinates

$n \geq 3$

$$\int_{B(0,\varepsilon)} |\Phi(y)|\, dy = C' \int_{B(0,\varepsilon)} \frac{1}{|x|^{n-2}}\, dx = C' \int_0^\varepsilon \frac{1}{r^{n-2}} r^{n-1}\, dr = C\varepsilon^2$$

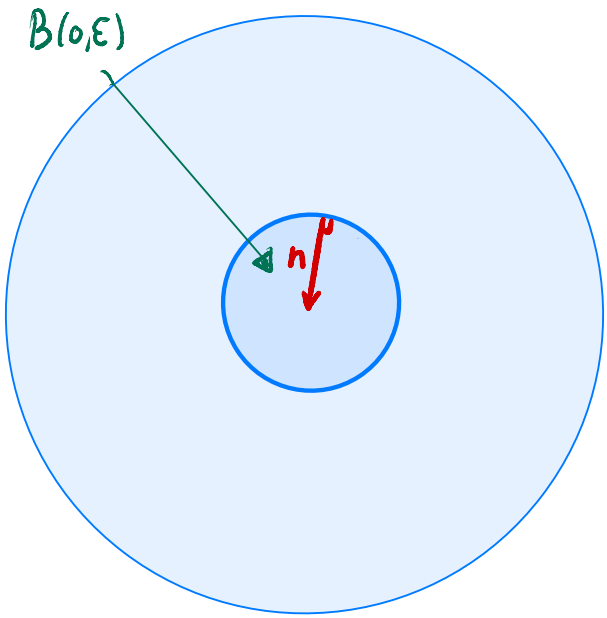
Therefore

$$|I_\varepsilon| = \left| \int_{B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy \right| \leq \int_{B(0,\varepsilon)} |\Phi(y)|\, |\Delta_x f(x-y)|\, dy$$

$$\leq C' \|\Delta_x f\|_{L^\infty(\mathbb{R}^n)} \begin{cases} \varepsilon^2 \log(\varepsilon) & n=2 \\ \varepsilon^2 & n \geq 3 \end{cases} \longrightarrow 0 \text{ as } \varepsilon \to 0$$

Estimate for J_ε

Support of f is inside $\square$



$$\Delta_x f(x-y) = \Delta_y f(x-y)$$

$$\Downarrow$$

$$J_\varepsilon = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\, \Delta_y f(x-y)\, dy$$

integration by parts

$$= -\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} D\Phi(y) \cdot D_y f(x-y)\, dy \longleftarrow K_\varepsilon$$

$$+ \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial n}(x-y)\, dS(y) \longleftarrow L_\varepsilon$$

$$y \in \partial B(0,\varepsilon) \quad \Phi(y) = C' \begin{cases} \log(\varepsilon) & n=2 \\ \left(\frac{1}{\varepsilon}\right)^{n-2} & n \geq 3 \end{cases}$$

$$|L_\varepsilon| \leq C |\Phi(\varepsilon)|\, \|Df\|_{L^\infty(\mathbb{R}^n)} (\text{area of } \partial B(0,\varepsilon))$$

$$\leq C' \varepsilon^{n-1} \begin{cases} \log(\varepsilon) & n=2 \\ \frac{1}{\varepsilon^{n-2}} & n \geq 3 \end{cases} \longrightarrow 0 \text{ as } \varepsilon \longrightarrow \varepsilon$$

integration by parts

$$K_\varepsilon = \underbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta\Phi(y)\, f(x-y)\, dy}_{=0} - \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial n}(y)\, f(x-y)\, dy$$

since $\Delta\Phi(y) = 0$ here!

Recall: $n=2$ $\Phi(y) = -\frac{1}{2\pi}\log(|y|)$

$$D\Phi(y) = -\frac{1}{2\pi}\frac{1}{|y|}\frac{y}{|y|} = -\frac{1}{2\alpha(2)}\frac{y}{|y|^2}$$

$n \geq 3$ $\Phi(y) = \frac{1}{n(n-2)\alpha(n)}\frac{1}{|y|^{n-2}}$

$$D\Phi(y) = \frac{-(n-2)}{n(n-2)\alpha(n)}\frac{1}{|y|^{n-1}}\frac{y}{|y|}$$

$$= \frac{-1}{n\alpha(n)}\frac{y}{|y|^n}$$

So for all n:

$$D\Phi(y) = \frac{-1}{n\alpha(n)}\frac{y}{|y|^n}$$

On $\partial B(0,\varepsilon)$: $n = -\frac{y}{\varepsilon}$, $D\Phi(y) = \frac{-1}{n\alpha(n)}\frac{y}{\varepsilon^n}$

so that:

$$n \cdot D\Phi(y) = \frac{|y|^2}{n\alpha(n)}\frac{1}{\varepsilon^{n+1}} = \frac{1}{n\alpha(n)\varepsilon^{n-1}}$$

$$= \frac{1}{\text{area of}(\partial B(0,\varepsilon))} \; !$$

We have that:

$$K_\varepsilon = -\int_{\partial B(0,\varepsilon)} D\Phi(y)\cdot n \, f(x-y)\, dS(y)$$

$$= -\int_{\partial B(0,\varepsilon)} \frac{1}{n\alpha(n)\varepsilon^{n-1}} f(x-y)\, dy$$

$$= \frac{-1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y)\,dy$$ ← This is the average of f in the spherical shell $\partial B(0,\varepsilon)$

$$=: \fint_{\partial B(0,\varepsilon)} f(x-y)\,dy \longrightarrow f(x) \text{ as } \varepsilon \to 0$$

since f is continuous.

Let $\varepsilon \to 0$ to conclude that $\Delta u(x) = -f(x)$ ∎

MEAN VALUE FORMULAS

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u harmonic, $\Delta u = 0$ (in U)

Define $$\varphi(r) = \fint_{\partial B(x,r)} u(y)\,dS(y)$$

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u(y)\,dS(y)$$ $y = x + rz$, $dS(y) = r^{n-1}\,dS(z)$, $z \in \partial B(0,1)$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz)\,dS(z).$$

Then

$$\varphi'(r) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} Du(x+rz)\cdot z\,dS(z)$$ change back to y

$$= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} Du(y)\cdot \mathbf{n}\,dS(y)$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy = 0.$$

Therefore: $\varphi(r) = \lim_{r \to 0} \varphi(r) = \lim_{r \to 0} \int_{\partial B(x,r)} u(y)\, dS(y)$

$= u(x)$ Since u is continuous.

Conclude:

> u harmonic in U, $B(x,r) \subset U$
> $\Rightarrow u(x) = \fint_{\partial B(x,r)} u(x)\, dS(y)$

MEAN VALUE FORMULA

Assume now that u satisfies the mean value property, and that u is not harmonic.

Then there must be a ball $B(x,r)$ on which $\Delta u(y) > 0$.

But

$$0 = \varphi'(r) = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy > 0. \quad !$$

$0 = \varphi'(r)$: mean value property

> 0: contradiction.

> $u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y) \Rightarrow u$ harmonic

MEAN VALUE FORMULA

$$\int_{B(x,r)} u(y)\, dy = \int_0^r \left(\int_{\partial B(x,s)} u(y)\, dS(y) \right) ds = u(x) \int_0^r n\alpha(n) s^{n-1}\, ds$$

$$= u(x)\, \alpha(n) r^n.$$