so that:

$$\Delta u = 0 \iff u(x) = \fint_{B(x,r)} u(y)\,dy$$

MEAN VALUE FORMULA

Maximum principle

Theorem. u harmonic in U, $u \in C(\bar{U}) \cap C^2(U)$.

a) $\max_{\bar{U}} u = \max_{\partial U} u$

b) if $u(x_0) = \max_{\bar{U}} u$ and $x_0 \in U$, then $u(x) = u(x_0)$ for all $x \in \bar{U}$.

Proof. $M = \max_{\bar{U}} u$. Assume $u(x_0) = M$ for some $x_0 \in U$. Let $r < \operatorname{dist}(x_0, \partial U)$.

$$M = u(x_0) = \fint_{B(x_0,r)} u\,dx \leq M \Rightarrow u(x) = M \text{ for all } x \in B(x_0, r).$$

Thus the set $\{x \in U \mid u(x) = M\}$ is open in U, since u is continuous it is also relatively closed. Since U is connected it is either empty or equal to U.

This proves b) $\Rightarrow$ a).

∎

Estimating the derivatives

Let η^ε be the standard mollifier.

$$u_\varepsilon(x) = \int_{B(x,\varepsilon)} \eta^\varepsilon(x-y)\, u(y)\, dy \qquad u_\varepsilon \text{ is } C^\infty(U^\varepsilon)$$

$$= \int_{B(x,\varepsilon)} \frac{1}{\varepsilon^n} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\, dy$$

polar coordinates

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \left(\int_{\partial B(x,r)} u(y)\, dS(y) \right) dr$$

mean value property

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) u(x)\, n\alpha(n)\, r^{n-1}\, dr$$

$$= \frac{u(x)}{\varepsilon^n} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \int_{\partial B(x,r)} dS(y)\, dr$$

polar to Cartesian coordinates

$$= u(x) \int_{B(x,\varepsilon)} \eta^\varepsilon(x)\, dx = u(x) \quad !$$

$$\Rightarrow \boxed{u \in C^\infty(U)}$$

We bound the derivatives of u.

α multi-index $|\alpha| = k \geq 0$

u harmonic in U, $x_0 \in U$, $B(x_0, r) \subset U$.

A) $$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

B) $$C_0 = \frac{1}{\alpha(n)} \qquad C_k = \frac{(2^{n+1} n k)^k}{\alpha(n)}, \quad k = 1, 2, 3, \ldots$$

Proof:

$k = 0$. $$|u(x_0)| = \left| \fint_{B(x_0,r)} u \, dx \right| \leq \frac{1}{\alpha(n)} \frac{1}{r^n} \int_{B(x_0,r)} |u| \, dx = \frac{1}{\alpha(n)} \frac{1}{r^n} \|u\|_{L^1(B(x_0,r))}$$

$k = 1$

$D^\alpha u = u_{x_i}$, observe that u_{x_i} is harmonic.

$$|u_{x_i}(x_0)| = \left| \frac{2^n}{\alpha(n) r^n} \int_{B(x_0, r/2)} u_{x_i} \, dx \right| = \frac{2^n}{\alpha(n) r^n} \left| \int_{\partial B(x_0, r/2)} u n_i \, dS \right|$$

↑ integration by parts: reduce the number of derivatives

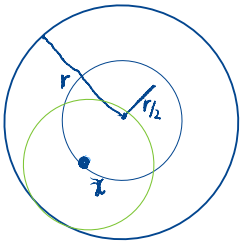
$$\leq \frac{2^n}{\alpha(n) r^n} n \alpha(n) \frac{r^{n-1}}{2^{n-1}} \|u\|_{L^\infty(\partial B(x_0, r/2))}$$

$$= \frac{2n}{r} \|u\|_{L^\infty(\partial B(x_0, r/2))} \quad (*)$$

$x \in \partial B(x_0, r/2)$

$\Rightarrow B(x, r/2) \subset B(x_0, r) \subset U$.

use estimate for $k = 0$ on $B(x, r/2)$



$$|u(x)| \le \frac{1}{\alpha(n)} \frac{2^n}{r^n} \|u\|_{L^1(B(x,r/2))}$$

$$\Rightarrow \|u\|_{L^\infty(\partial B(x_0,r/2))} \le \frac{2^n}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,r))}$$

Insert this in (★)

$$|u_{x_i}(x_0)| \le \frac{2n}{r} \frac{2^n}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,r))}$$

$$= \frac{2^{n+1} n}{\alpha(n)} \frac{1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))}$$

this shows the estimate for $|\alpha| = 1$.

Induction step

Assume bound holds for k. $|\alpha| = k+1$, $\alpha = \beta + e_i$.

$D^\alpha u$ is harmonic.

$$|D^\alpha u(x_0)| = |D^\beta u_{x_i}(x_0)| = \left| \frac{(k+1)^n}{\alpha(n) r^n} \int_{B(x_0, r/k+1)} D^\beta u_{x_i} \, dx \right|$$

$$\le \frac{(k+1)^n}{\alpha(n) r^n} \int_{\partial B(x_0, r/k+1)} |D^\beta u \, n_i| \, dS$$

$$\le \frac{(k+1)^n}{\alpha(n) r^n} n \alpha(n) \frac{r^{n-1}}{(k+1)^{n-1}} \|D^\beta u\|_{L^\infty(\partial B(x_0, r/k+1))}$$

(★)

$$= \frac{(k+1) n}{r} \|D^\beta u\|_{L^\infty(\partial B(x_0, r/k+1))}. \quad (★)$$

$x \in \partial B(x_0, r/k+1)$

$$\Rightarrow B\left(x, \frac{k}{k+1} r\right) \subset B(x_0, r) \subset U.$$

induction hypothesis

$$|D^\beta u(x)| \le \frac{(2^{n+1} n k)^k}{\alpha(n)} \frac{(k+1)^{n+k}}{k^{n+k}} \frac{1}{r^{n+k}} \|u\|_{L^1(B(x, \frac{k}{k+1} r))}$$

By (★)

$$|D^\alpha u(x_0)| \le \frac{(k+1)n}{r} \frac{(2^{n+1})^k n^k k^k (k+1)^{n+k}}{\alpha(n) \; k^{n+k} \; r^{n+k}} \|u\|_{L^1(B(x_0, r))}$$

$$= \frac{(2^{n+1})^k n^{k+1} (k+1)^{k+1}}{\alpha(n) r^{n+k+1}} \left(\frac{k+1}{k}\right)^n \|u\|_{L^1(B(x_0,r))}$$

$$\left(\frac{k+1}{k}\right)^n \le 2^{n+1}$$

$$\le \frac{(2^{n+1} n(k+1))^{k+1}}{\alpha(n)} \frac{1}{r^{n+k+1}} \|u\|_{L^1(B(x_0,r))}$$

Liouville's theorem

Suppose $u: \mathbb{R}^n \to \mathbb{R}$ is bounded and harmonic. Then u is constant.

Proof:

$|Du|^2 = \sum_{i=1}^n u_{x_i}^2$, use previous bound on u_{x_i}.

to get

$$|Du(x_0)| \le \frac{\sqrt{n} C_1}{r^{n+1}} \int_{B(x_0,r)} |u| \, dx \le \frac{\sqrt{n} C_1 r^n \alpha(n)}{r^{n+1}} \|u\|_{L^\infty(\mathbb{R}^n)} \to 0 \text{ as } r \to \infty$$

$\Rightarrow Du = 0 \Rightarrow u$ is a constant. ∎