Harmonic in U $\Rightarrow$ analytic in U

$r = \frac{1}{4}\operatorname{dist}(x_0, \partial U)$ $\qquad M := \frac{1}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,r))}$

$x \in B(x_0, r) \Rightarrow B(x, r) \subset U$

$$|D^\alpha u(x)| \leq \frac{(2^{n+1} n|\alpha|)^{|\alpha|}}{\alpha(n)} \frac{1}{r^{|\alpha|+n}} \|u\|_{L^1(B(x,r))}$$

$$\leq M \left(\frac{2^{n+1} n}{r}\right)^{|\alpha|} |\alpha|^{|\alpha|}$$

this gives an L^∞ bound since RHS indep. on x.

Facts: $|\alpha|^{|\alpha|} \leq e^{|\alpha|} |\alpha|!$

$$n^k = (1 + \ldots + 1)^k = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} \Rightarrow |\alpha|! \leq n^{|\alpha|} \alpha! \; ; |\alpha| = k$$

$$\Rightarrow |\alpha|^{|\alpha|} \leq n^{|\alpha|} \alpha! \, e^{|\alpha|}$$

so $$\boxed{\|D^\alpha u\|_{L^\infty(B(x_0,r))} \leq M \left(\frac{2^{n+1} n^2 e}{r}\right)^{|\alpha|} \alpha!}$$

Taylor series with remainder

$$u(x) = \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha u(x_0) (x - x_0)^\alpha$$

$$+ \sum_{|\alpha|=N} \frac{1}{\alpha!} D^\alpha u(x_0 + t(x - x_0)) (x - x_0)^\alpha, \quad t \in (0,1).$$

↑ Remainder term. R_N

must show that $R_N \to 0$ as $N \to \infty$

$$|R_N(x)| \le \sum_{|\alpha|=N} M\left(\frac{2^{n+1} n^2 \ell}{r}\right)^{|\alpha|} |(x-x_0)^\alpha|$$

choose $|x-x_0| \le \frac{r}{2^{n+2} n^3 \ell}$

$$\le \sum_{|\alpha|=N} M\left(\frac{2^{n+1} n^2 \ell}{r}\right)^{|\alpha|} \left(\frac{r}{2^{n+2} n^3 \ell}\right)^{|\alpha|}$$

$$\le \sum_{|\alpha|=N} M\left(\frac{2^{n+1} n^2 \ell}{r}\right)^{|\alpha|} \left(\frac{r}{2^{n+2} n^3 \ell}\right)^{|\alpha|}$$

$$= CM\, n^N \frac{1}{(2n)^N} = CM \frac{1}{2^N} \to 0$$

Fact: There are $\binom{N+n-1}{N}$ derivatives of order N

$$\frac{(N+n-1)!}{N!\,(n-1)!} \le C\, n^N$$

$\ell! \approx \ell^{\ell+1/2}$

$$\sum_{|\alpha|=N} = \binom{N+n-1}{N} \le C n^N \leftarrow \text{fact}$$