The heat equation

(H) $u_t - \Delta u = 0$ homogeneous

$u_t - \Delta u = f$ non-homogeneous

} in $\mathbb{R}^n \times [0,\infty)$

often coupled to initial data

$$u|_{t=0} = g$$

Fundamental solution

If u solves (H), set $v(x,t) = u(\lambda x, \lambda^2 t)$, $\lambda > 0$.

$$\left.\begin{aligned} v_t(x,t) &= \lambda^2 u_t(\lambda x, \lambda^2 t) \\ \Delta v(x,t) &= \lambda^2 \Delta u(\lambda x, \lambda^2 t) \end{aligned}\right\} \Rightarrow v_t - \Delta v = 0.$$

Look for a solution on the form

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$$

$$u_t = \frac{-\alpha}{t^{\alpha+1}} v\left(\frac{x}{t^\beta}\right) - \frac{\beta}{t^{\alpha+1+\beta}} Dv\left(\frac{x}{t^\beta}\right) \cdot x$$

$$\Delta u = \frac{1}{t^{\alpha+2\beta}} \Delta v\left(\frac{x}{t^\beta}\right) \qquad y = \frac{x}{t^\beta}$$

$$\beta = 1/2$$

$$0 = u_t - \Delta u = \frac{-1}{t^{\alpha+1}}\left(\alpha v(y) + \frac{1}{2} Dv(y)\cdot y + \Delta v(y)\right)$$

Set $v(y) = w(|y|)$

$$v_{y_i y_i} = \left(w'(r)\frac{y_i}{|y|}\right)_{y_i} = w''(r)\frac{y_i^2}{|y|^2} + w'(r)\left(\frac{|y| - y_i^2/|y|}{|y|^2}\right)$$

$$= w''(r)\frac{y_i^2}{|y|^2} + w'(r)\left(\frac{1 - y_i^2/|y|^2}{|y|}\right)$$

$$\Rightarrow \underline{\Delta v(|y|) = w''(r) + w'(r)\frac{n-1}{r}}.$$

$$\underline{Dv(y)\cdot y = \sum_{i=1}^{n} w'(r)\frac{y_i^2}{|y|} = w'(r)\, r}$$

Therefore: (setting $\alpha = \frac{n}{2}$)

$$\frac{n}{2} w(r) + \frac{1}{2} r w'(r) + w''(r) + w'(r)\left(\frac{n-1}{r}\right) = 0 \quad \Big| \, r^{n-1}$$

$$\frac{1}{2}\underbrace{\left(n r^{n-1} w(r) + r^n w'(r)\right)}_{(r^n w(r))'} + \underbrace{\left(r^{n-1} w''(r) + (n-1) r^{n-2} w'(r)\right)}_{(r^{n-1} w'(r))'} = 0$$

we integrate to get:

$$\frac{1}{2} r^n w(r) + r^{n-1} w'(r) = A$$ ← constant

"boundary" conditions at $r = \infty$ s.t. $A = 0$

$$\frac{w'(r)}{w(r)} = \frac{1}{2} r \quad \Rightarrow \quad w(r) = B e^{-\frac{r^2}{4}}$$

Define:

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$$

$t > 0$!

fundamental solution

Heat kernel

a) $\Phi_t - \Delta \Phi = 0 \qquad x \in \mathbb{R}^n, \ t > 0$

b) $$\int_{\mathbb{R}^n} \Phi(x,t)\,dx = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x|^2/4t}\,dx$$ change of variables

$$= \frac{1}{(4\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2}\,dz = 1, \quad t > 0$$

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-s^2}\,ds$$

c) $\Phi \in C^\infty(\mathbb{R}^n \times (0,\infty))$