The initial value problem (Cauchy problem)

$$(P) \quad \begin{cases} u_t - \Delta u = 0 & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = g(x) \end{cases}$$

We assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

Theorem

The function

$$u(x,t) := \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy$$

(i) is in $C^\infty(\mathbb{R}^n \times (0,\infty))$

(ii) $u_t - \Delta u = 0$ for $(x,t) \in \mathbb{R}^n \times (0,\infty)$

(iii) $\lim_{\substack{(x,t)\to(x_0,0) \\ t>0}} u(x,t) = g(x_0)$ pointwise in x_0

Proof.

(i) The function $\Phi(x,t)$ is C^∞, with (uniformly) bounded derivatives of all orders in the set:

$$\mathbb{R}^n \times [\delta, \infty) \quad \forall\, \delta > 0$$

(ii) $u_t - \Delta u = \int_{\mathbb{R}^n} \overbrace{[\Phi_t - \Delta\Phi](x-y,t)}^{0} g(y)\,dy = 0$

(iii) g is continuous, $\forall$ given ε choose δ s.t.

$$|y - x_0| < \delta \Rightarrow |g(y) - g(x_0)| < \varepsilon$$

If $|x - x_0| < \delta/2$, then

$$|u(x,t) - g(x_0)| \le \int_{\mathbb{R}^n} \Phi(x-y,t)\,|g(y) - g(x_0)|\,dy$$

$$\le \left(\int_{B(x_0,\delta)} + \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \right) \Phi(x-y,t)\,|g(y) - g(x_0)|\,dy$$

$$= I + J$$

We have that $I \le \varepsilon$, since $|y - x_0| < \delta$ here.

To evaluate J, note that $|y-x| \ge \delta$ and $|x - x_0| \le \delta/2$

$\Rightarrow$

$$|y - x_0| \le |y-x| + |x - x_0| \le |y-x| + \delta/2 \le |y-x| + \tfrac{1}{2}|y - x_0|$$

$$|y-x| \ge \tfrac{1}{2}|y - x_0|$$

$\Rightarrow$ $e^{-|y-x|^2/4t} \le e^{-|y-x_0|^2/16t}$

Therefore:

$$J \le C \|g\|_{L^\infty} t^{-n/2} \int_{|y-x_0|>\delta} e^{-\frac{|y-x_0|^2}{16t}} dy$$

Change of variables: $z = \frac{|y-x_0|}{\sqrt{t}}$

$$t^{n/2} dz = dy$$

$$= C \|g\|_{L^\infty} \int_{|z| \ge \delta/\sqrt{t}} e^{-|z|^2/16} dz \longrightarrow 0 \quad \text{as } t \to 0^+.$$

■

Duhamel's principle

$x \in \mathbb{R}$ $x = x(t)$.

$$\begin{cases} x' - ax = 0 & t > 0 \\ x(0) = x_0 \end{cases}$$

Solving by multiplying with e^{-at}

$$\underbrace{e^{-at} x' - a e^{-at} x}_{} = 0$$

$$\frac{d}{dt}(e^{-at} x) = 0 \Rightarrow x(t) = x_0 e^{at}.$$

Non-homogeneous $x' - ax = f(t) \Rightarrow x(t) = x_0 e^{at} + \int_0^t e^{a(t-s)} f(s) ds$

Duhamel's principle

Consider now the abstract linear differential equation:

$$\begin{cases} X' - AX = 0 & t > 0 \\ X(0) = X_0 \end{cases}$$

Assume that we have a flow-map φ_t^A such that

a) $\varphi_t^A(X_0) = X(t)$

b) $\frac{d}{dt}\varphi_t^A - A\varphi_t^A = 0$

c) $\varphi_{t+s}^A = \varphi_t^A \circ \varphi_s^A$

d) φ_t^A is linear

We wish to solve the non-homogeneous problem:

$$X'(t) - AX(t) = F(t)$$

Apply the linear operator φ_{-t}^A to both sides

$$\underbrace{\varphi_{-t}^A X'(t) - A\varphi_{-t}^A X(t)}_{\text{Leibnitz}} = \varphi_{-t}^A F(t)$$

$$\frac{d}{dt}\left(\varphi_{-t}^A X(t)\right) = \varphi_{-t}^A F(t)$$

Integrate over $(0,t)$

$$\varphi_{-t}^{A} X(t) - \varphi_{0}^{A} X_0 = \int_0^t \varphi_{-s}^{A} F(s)\,ds$$

$$\varphi_{-t}^{A} X(t) = X_0 + \int_0^t \varphi_{-s}^{A} F(s)\,ds$$

Apply φ_t^A

$$\boxed{X(t) = \varphi_t^A X_0 + \int_0^t \varphi_{t-s}^{A} F(s)\,ds}$$

Duhamel's method.

For the heat equation:

$$A = \Delta$$

$$\varphi_t^A(g)(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\,dy$$

the non-homogeneous problem

$$\begin{cases} u_t - \Delta u = f & t>0,\ x \in \mathbb{R}^n \\ u(x,0) = g(x) \end{cases} \qquad f = f(x,t).$$

Duhamel's method gives the solution:

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(y-x,t)\, g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} \Phi(y-x,t-s)\, f(y,s)\,dy\,ds$$

↑

Must check!!

Why?

Enough to check this for $f \in C_1^2(\mathbb{R}^n \times (0,\infty))$

Claim

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\, dy\, ds$$

satisfies $\begin{cases} u_t - \Delta u = f & t>0 \\ u = 0 & t=0 \end{cases}$

Assume f has compact support

$$u_t(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y, t-s)\, dy\, ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy$$

$$u_{x_i x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_{x_i x_j}(x-y, t-s)\, dy\, ds$$

Both continuous since $f \in C_1^2$

$$u_t - \Delta u = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\left(f_t(x-y,t-s) - \Delta_x f(x-y,t-s)\right) dy\, ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy$$

$\downarrow$ $\varepsilon > 0$

$$= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,t-s)\,dy\,ds \quad =: I_\varepsilon$$

$$+ \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,t-s)\,dy\,ds \quad =: J_\varepsilon$$

$$+ \int_{\mathbb{R}^n} \Phi(y,t) f(y,0)\,dy. \qquad =: K_\varepsilon$$

$$|J_\varepsilon| \le \left(\|f_t\|_{L^\infty} + \|\Delta f\|_{L^\infty}\right) \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\,dy\,ds \le C\varepsilon.$$

$$I_\varepsilon = \int_\varepsilon^t \int_{\mathbb{R}^n} \overbrace{\left(\frac{\partial}{\partial s} - \Delta_y\right)\Phi(y,s)}^{=0} f(x-y,t-s)\,dy\,ds$$

$$\underbrace{-\int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy}_{-K} + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy$$

$$u_t - \Delta u = \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy$$

$$\downarrow \varepsilon \to 0^+$$

$$= f(x,t).$$

To show that $u(x,0) = 0$, observe

$$|u(x,t)| \le \|f\|_{L^\infty} \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\,dy\,ds \le \|f\|_{L^\infty} t$$

Let $t \to 0^+$.

This proves the claim.