

We consider Hamilton-Jacobi eq.

$$\left. \begin{aligned} u_t + H(Du, x) &= 0 && \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g && \text{on } \mathbb{R}^n \times \{0\} \end{aligned} \right\} (1)$$

given  $g \in C(\mathbb{R}^n)$  and  $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$ .

Main challenge: classical  $C^1$  solutions to (1) may not exist. Need to relax definition of solution.

Recall

§3.3 considers  $H(Du)$  and proves existence and uniqueness of weak solutions.

Plan

- Introduce new type of solution (viscosity solutions)
- Prove existence and uniqueness of such solutions.

Def A bounded and uniformly continuous  
 $u: \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$  satisfying

i)  $u(\cdot, 0) = g$

ii) for each  $v \in C^\infty(\mathbb{R}^n \times (0, \infty))$

if  $u - v$  has local max (min) at  $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ ,

then  $v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \leq 0$  ( $\geq 0$ ),

is called a viscosity solution of (1).

Motivation For  $\varepsilon > 0$ , one can often

approximate (1) by PDE

$$u_t^\varepsilon + H(Du^\varepsilon, x) = \underbrace{\varepsilon \Delta u^\varepsilon}_{\text{viscosity}} \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

$$u^\varepsilon = g \quad \text{on } \mathbb{R}^n \times \{0\},$$

with smooth solution  $u^\varepsilon$  on  $\mathbb{R}^n \times (0, \infty)$ .

And one can find subsequence  $\{u^{\varepsilon_j}\}_{j=1}^\infty$  and  
 $u \in C(\mathbb{R}^n \times [0, \infty))$  s.t.  $u^{\varepsilon_j} \rightarrow u$  locally uniformly  
in  $\mathbb{R}^n \times [0, \infty)$  as  $j \rightarrow \infty$  (and  $\varepsilon_j \rightarrow 0$ ).

Consider (i) and suppose  $u(x_0, t_0) - v(x_0, t_0)$  is a strict local max. Then for small  $\varepsilon_j$ , there exists an  $(x_j, t_j)$  close to  $(x_0, t_0)$  that is a strict local max of  $u^\varepsilon - v$  and  $(x_j, t_j) \rightarrow (x_0, t_0)$  as  $j \rightarrow \infty$ .

As  $u^\varepsilon$  is smooth

$$Du^{\varepsilon_j}(x_j, t_j) = Dv(x_j, t_j), \quad u_t^{\varepsilon_j}(x_j, t_j) = v_t(x_j, t_j)$$

$$\text{and max} \Rightarrow \frac{\partial^2}{\partial x_i^2} (u^{\varepsilon_j} - v)(x_j, t_j) \leq 0 \quad \forall i$$

$$\Rightarrow \Delta u^\varepsilon(x_j, t_j) \leq \Delta v(x_j, t_j)$$

$$\text{Hence, } v_t(x_j, t_j) + H(Dv(x_j, t_j), x_j)$$

$$= u_t^{\varepsilon_j}(x_j, t_j) + H(Du^{\varepsilon_j}(x_j, t_j), x_j)$$

$$= \varepsilon_j \Delta u^{\varepsilon_j}(x_j, t_j) \leq \varepsilon_j \Delta v(x_j, t_j)$$

Taking limit  $j \rightarrow \infty$  yields

$$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \leq 0.$$

If  $u-v$  instead has a nonstrict local max at  $(x_0, t_0)$ , then  $u-\tilde{v}$  with

$\tilde{v}(x, t) = v(x, t) + \delta(|x-x_0|^2 + (t-t_0)^2)$  has a strict local max at  $(x_0, t_0)$

$$\Rightarrow \underbrace{\tilde{v}_t(x_0, t_0) + H(D\tilde{v}(x_0, t_0), x_0)}_{= v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0)} \leq 0 \quad (\text{By construction}).$$

## Consistency of definition

Property 1 If  $u \in C^1(\mathbb{R}^n \times [0, \infty))$  is classical solution to (1), is bounded and uniformly cont., then  $u$  is a viscosity solution.

Proof If  $v \in C^\infty$  and  $u-v$  has local max at  $(x_0, t_0)$ , then

$$u_t = v_t, \quad Du = Dv \quad \text{at } (x_0, t_0), \text{ so,}$$

$$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) = u_t(x_0, t_0) + H(Du(x_0, t_0), x_0) = 0 \leq 0.$$

and similarly,  $V_t(x_0, t_0) + H(DV(x_0, t_0), x_0) = 0 \geq 0$   
 at a local min  $(x_0, t_0)$  of  $u - v$ .  $\square$

(The following lemma is used to show that consistency also holds for less regular solutions of (1).)

Lemma (Touching by  $C^1$  function)

Assume  $u \in C(\mathbb{R}^n)$  is continuous and differentiable at a point  $x_0$ . Then there exists a  $v \in C^1(\mathbb{R}^n)$  s.t.

$u(x_0) = v(x_0)$  and  
 $u - v$  has a strict local max at  $x_0$

Proof WLOG we may assume  $x_0 = 0$ ,  $u(0) = 0$ ,  $Du(0) = 0$  (else, consider  $\tilde{u}(x) = u(x + x_0) - u(x_0) + Du(x_0) \cdot x$ )

Near  $x = 0$ ,  $|u(x) - u(0)| = |Du(0) \cdot x| + o(|x|) = o(|x|)$

$\Rightarrow \frac{|u(x)|}{|x|} \rightarrow 0$  as  $x \rightarrow 0$ . So the function

$$\rho(x) = \begin{cases} \frac{|u(x)|}{|x|} & |x| \neq 0 \\ 0 & |x| = 0 \end{cases}$$

is continuous, and  $u(x) = |x| p(x)$ .

Let  $\mu(r) := \max_{x \in B(0,r)} |p(x)| \quad r \geq 0$ .

Then  $\mu$  is cont., nondecreasing and  $\mu(0) = 0$ .

Let  $v(x) := \int_{|x|}^{2|x|} \mu(r) dr + |x|^2 \quad x \in \mathbb{R}^n$

Then  $|v(x)| \leq |x| \mu(2|x|) + |x|^2 \Rightarrow$

$v(0) = 0$ , and  $|v_{x_i}(0)| = \lim_{h \rightarrow 0} \frac{|v(e_i h) - v(0)|}{|h|} = 0$

$\Rightarrow Dv(0) = 0$ .

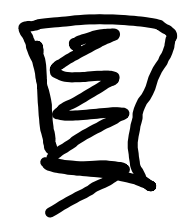
Further, for  $x \neq 0$ ,

$$Dv(x) = \mu(2|x|) \frac{2x}{|x|} - \mu(|x|) \frac{x}{|x|} + 2x$$

$\rightarrow 0$  as  $x \rightarrow 0$  so  $v \in C^1(\mathbb{R}^n)$ .

$$\begin{aligned} \text{Lastly, } u(x) - v(x) &= |x| p(x) - \int_{|x|}^{2|x|} \mu(r) dr - |x|^2 \\ &\leq |x| \mu(|x|) - |x| \mu(|x|) - |x|^2 \\ &= -|x|^2 \end{aligned}$$

$$\Rightarrow u(x) - v(x) = \begin{cases} 0 & \text{if } x=0 \\ < 0 & \text{if } x \neq 0. \end{cases}$$



Property 2: Let  $u$  be a viscosity solution of (1). Then at any point  $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ , where  $u$  is differentiable, it holds that

$$u_t(x_0, t_0) + H(Du(x_0, t_0), x_0) = 0$$

Proof By Lemma 1 on  $\mathbb{R}^n \times (0, \infty)$ ,  $\exists$  a  $C^1$  function  $V: \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$  s.t.  $u(x_0, t_0) = V(x_0, t_0)$  and  $u - V$  has a strict max at  $(x_0, t_0)$ .

For  $\varepsilon > 0$ , set  $V^\varepsilon(x, t) = \eta_\varepsilon * V(x, t)$

$$= \frac{C}{\varepsilon^n} \int_{\mathbb{R}^n \times (0, \infty)} V(y, s) \exp\left(\frac{1}{|x-y|^2 + |t-s|^2 - 1}\right) \frac{1}{B(x, t), 1} dy ds$$

with  $B((x, t), 1) \subset \mathbb{R}^{n+1}$ .

Then, locally in  $\mathbb{R}^{n+1}$ , e.g. near  $(x_0, t_0)$ ,  $V^\varepsilon \rightarrow V$ ,  $DV^\varepsilon \rightarrow DV$ ,  $V_t^\varepsilon \rightarrow V_t$  uniformly as  $\varepsilon \downarrow 0$ .

Hence,  $u - v^\varepsilon$  has a local max at some point  $(x_\varepsilon, t_\varepsilon)$  near  $(x_0, t_0)$ , and  $(x_\varepsilon, t_\varepsilon) \rightarrow (x_0, t_0)$  as  $\varepsilon \downarrow 0$ .

$u$  viscosity solution  $\Rightarrow$

$$V_t^\varepsilon(x_\varepsilon, t_\varepsilon) + H(DV^\varepsilon(x_\varepsilon, t_\varepsilon), x_\varepsilon) \leq 0$$

limit  $\varepsilon \downarrow 0$  and  $H$  cont.  $\Rightarrow$

$$V_t(x_0, t_0) + H(DV(x_0, t_0), x_0) \leq 0.$$

Further,  $u$  differentiable at strict max  $(x_0, t_0)$  of  $u - v$  implies  $u_t = v_t$  and  $Du = Dv$  at  $(x_0, t_0)$

so

$$u_t(x_0, t_0) + H(Du(x_0, t_0), x_0) \leq 0.$$

Next, Lemma 1 applied to  $-u \Rightarrow \exists \tilde{v} \in C^1$  s.t.  $-u - \tilde{v}$  has strict max at  $(x_0, t_0)$ .  
 $\Rightarrow u - \underbrace{(-\tilde{v})}_{\tilde{v}}$  has strict min at  $(x_0, t_0)$ ,  
 and, similarly as above,

$$u_t(x_0, t_0) + H(Du(x_0, t_0), x_0) \geq 0.$$

$\square$

## Uniqueness

We will restrict PDE to finite time window:

$$\left. \begin{aligned} u_t + H(Du, x) &= 0 && \text{in } \mathbb{R}^n \times (0, T] \\ u &= g && \text{on } \mathbb{R}^n \times \{0\}. \end{aligned} \right\} (2)$$

$g$  and  $H$  conf.

Def

$u: \mathbb{R}^n \times [0, T]$  is a viscosity sol of (2) if it is bounded, uniformly cont., and it holds that

i)  $u(\cdot, 0) = g$

ii) For any  $v \in C^0(\mathbb{R}^n \times [0, T])$ , if  $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$  is a local max (min) of  $u - v$ , then

$$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \leq 0 \quad (\geq 0).$$

Thm 1 applies here as well:

$u_t(x_0, t_0) + H(Du(x_0, t_0), x_0) = 0$  at any point  $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$  where  $u$  is differentiable.

And property ii) extends to max/min points  $(x_0, T)$ :

Lemma 2 If  $u$  is a viscosity solution of (2),  $v \in C^\infty(\mathbb{R}^n \times [0, T])$  and  $(x_0, T)$  is a local max (min) of  $u - v$ , then

$$V_t(x_0, T) + H(DV(x_0, T), x_0) \leq 0 \quad (\geq 0).$$

Proof As before, for  $\delta > 0$ ,  $\tilde{v}(x, t) = v(x, t) + \delta(|x - x_0|^2 + (t - T)^2)$  yields that  $u - \tilde{v}$  has a strict local max at  $(x_0, T)$ .

For  $\varepsilon > 0$ , let  $v^\varepsilon(x, t) := \tilde{v}(x, t) + \frac{\varepsilon}{T - t}$

for  $(x, t) \in \mathbb{R}^n \times (0, T)$ . Then for small  $\varepsilon > 0$ ,  $\exists$

$(x_\varepsilon, t_\varepsilon) \in \mathbb{R}^n \times (0, T)$  near  $(x_0, T)$  that is a local max of  $u - v^\varepsilon$ , and  $(x_\varepsilon, t_\varepsilon) \rightarrow (x_0, T)$  as  $\varepsilon \downarrow 0$ .

$$\Rightarrow V_t^\varepsilon(x_\varepsilon, t_\varepsilon) + H(DV^\varepsilon(x_\varepsilon, t_\varepsilon), x_\varepsilon) \leq 0$$

limit  $\varepsilon \downarrow 0$  &  $H$  cont. yields

$$\tilde{V}_t(x_0, T) + H(D\tilde{v}(x_0, T), x_0) \leq 0$$

and  $V_t(x_0, T) = \tilde{V}_t(x_0, T)$ ,  $DV(x_0, T) = D\tilde{v}(x_0, T)$ .

Local min ineq proved similarly. □

Next lectures:

- If  $H$  Lipschitz, then (2) has at most one viscosity solution.
- Existence of visc. solutions via control theory.