

Important points

- Energy methods heat eq, wave eq.
 - Mean-value formulas Laplace, heat eq.
 - Maximum principles formulas
 - Solution methods, fundamental solutions.
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- MoC, system of ODE, local solutions
 - Legendre transform, Hopf-Lax - HJ eq.
 - Lax-Oleinik formula, entropy condition, Rankine-Hugoniot.
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- $u_t + H(Du, x) = 0$

Viscosity solutions (be able to verify)

- Optimal control formulation, connection value function - HJB eq.
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Exercises 2019 1b, 2, 3, 4, 5. 2020 2, 3, 2022 5

2023 1a, 2, 3. Exercises on HJ eq and viscosity solutions. Ex week 47 Problem 3.

Duhamel's principle nonhomogeneous PDE

Transport

$$u_t + b \cdot Du = f(x, t)$$

$$u(x, 0) = 0$$

$$V_t(x, t; s) + b \cdot DV(x, t; s) = 0 \quad t > s, \quad x \in \mathbb{R}^n$$

$$V(x, s; s) = f(x, s)$$

$$u(x, t) = \int_0^t V(x, t; s) ds$$

Heat eq.

$$u_t - \Delta u = f(x, t)$$

$$u(x, 0) = 0$$

$$V_t(x, t; s) - \Delta V(x, t; s) = 0 \quad t > s, \quad x \in \mathbb{R}^n$$

$$V(x, s; s) = f(x, s)$$

$$u(x, t) = \int_0^t V(x, t; s) ds$$

Wave eq 1D

$$u_{tt} - u_{xx} = f(x, t)$$

$$u(x, 0) = 0, u_t(x, 0) = 0$$

$$v_{tt}(x, t; s) - v_{xx}(x, t; s) = 0 \quad t > s, x \in \mathbb{R}$$

$$v_t(x, s; s) = f(x, s), v(x, s; s) = 0$$

$$u(x, t) = \int_0^t v(x, t; s) ds$$

2019 1 b) Either find solution with Noc and verify it is a solution

OR explain why PDE cannot be solved.
Either way draw some projected characteristics.

$$t u_t + 2u_x = 0$$

$$x \in \mathbb{R}, t \geq 0$$

$$u(x, 0) = \sin(x)$$

$$x \in \mathbb{R}$$

$$\Gamma = \{(x, 0) \mid x \in \mathbb{R}\}$$

Let us write $x_2 u_{x_2} + 2u_{x_1} = 0 \quad x \in \mathbb{R}, t \geq 0$
 $u(x_1, 0) = \sin(x_1) \quad x_1 \in \mathbb{R}$

$$\Gamma = \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$$

$$Du = \begin{bmatrix} u_{x_1} \\ u_{x_2} \end{bmatrix}, \quad F(Du, x) = x_2 u_{x_2} + 2u_{x_1} = 0$$

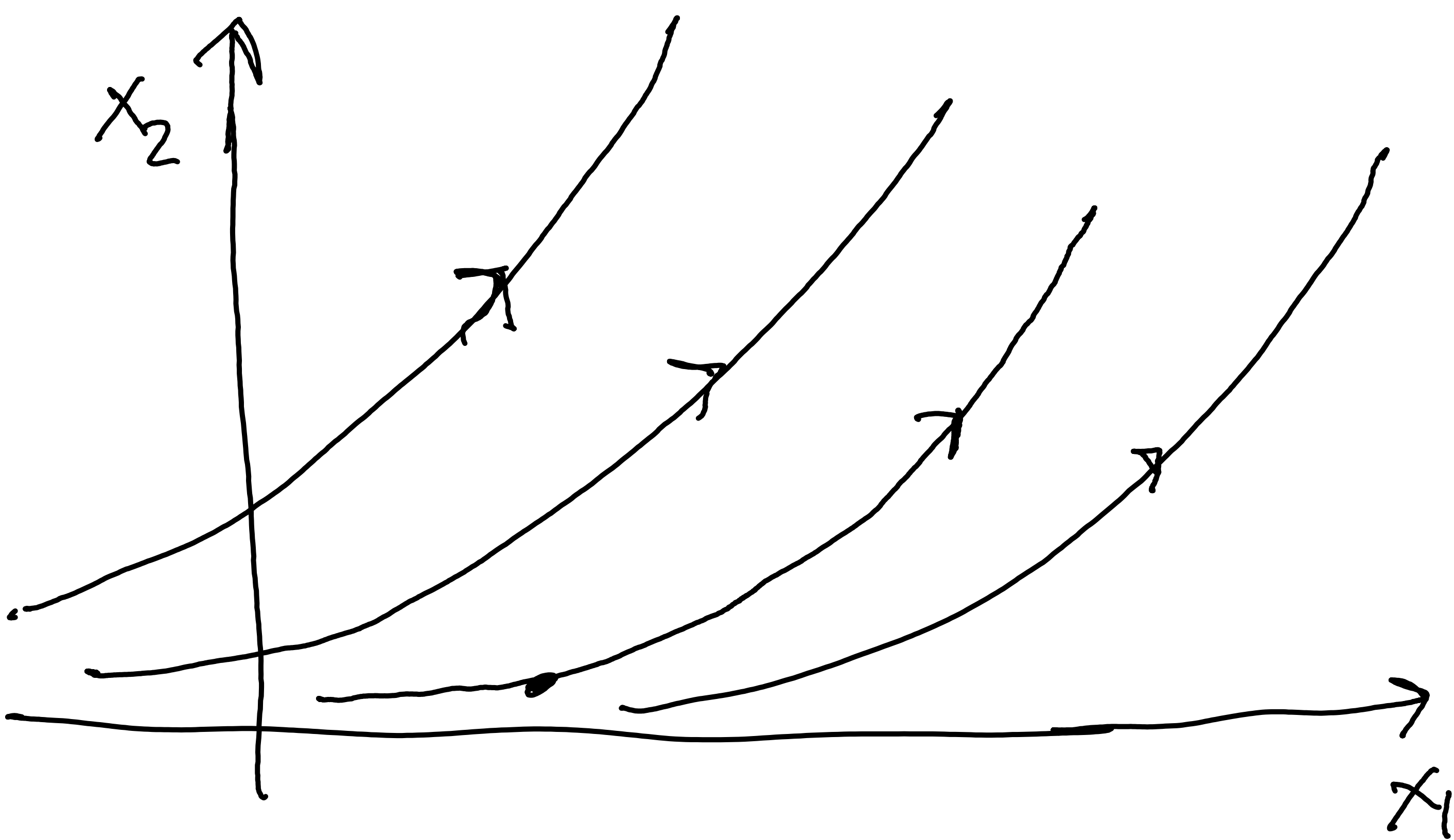
$$F(p, x) = x_2 p_2 + 2p_1 = 0$$

$$\dot{\underline{X}} = D_p F = \begin{bmatrix} 2 \\ x_2 \end{bmatrix} \Rightarrow \begin{aligned} \underline{X}_1(t) &= 2t + x_0^1 \\ \underline{X}_2(t) &= x_0^2 e^t = 0 \end{aligned}$$

Noncharacteristic BC $\nabla(x_0^1) \cdot \dot{\underline{X}}(0) \neq 0$ does not hold as $\nabla(x_0^1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \perp \dot{\underline{X}}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

So a local solution does not exist near Γ .

Characteristics $\underline{X}(t)$ if $x_0^2 > 0$.



2019 [2] Let $T \geq 0$. Find the general solution of the backwards problem

$$u_{tt} = u_{xx} \quad x \in \mathbb{R}, \quad t \in (0, T)$$

$$u(x, T) = g(x)$$

$$u_t(x, T) = h(x)$$

Is the solution unique?

Sol Consider $v(x, t) := u(x, T-t)$.

Then $v_t = -u_t$, $v_{tt} = u_{tt}$, and $v_{xx} = u_{xx}$ so

$$v_{tt} = v_{xx} \quad x \in \mathbb{R}, \quad t \in (0, T)$$

$$v_t(x, 0) = -h(x)$$

$$v(x, 0) = g(x)$$

d'Alembert's formula:

$$v(x, t) = \frac{f(x+t) + g(x-t)}{2} - \frac{1}{2} \int_{x-t}^{x+t} h(\gamma) d\gamma$$

$\Rightarrow$

$h \in C^2, g \in C^1$, formula is unique C^2 -sol
of forward PDE $\Rightarrow$

$$\begin{aligned} u(x, t) &= v(x, T-t) \\ &= \frac{g(x+(T-t)) + g(x-(T-t))}{2} \\ &\quad - \frac{1}{2} \int_{x-(T-t)}^{x+(T-t)} h(y) dy \end{aligned}$$

is unique sol of backward problem.

2019 [3] changed problem. Find an entropy solution to the problem

$$\left. \begin{aligned} u_t + f(u)_x &= 0 & x \in \mathbb{R}, t \geq 0 \\ u(x, 0) &= \begin{cases} 3 & x < 0 \\ 1 & x > 0 \end{cases} \end{aligned} \right\} (1)$$

where $f(u) = u^2$. Verify that the obtained solution satisfies the Rankine-Hugoniot condition.

$u = 2_x v$ where v solves HJ eq.

$$v_t + (v)_x^2 = 0$$

$$v(x,0) = \underbrace{\int_0^x u(x,0) dx}_{g(x)} = \begin{cases} 3x & x < 0 \\ x & x > 0 \end{cases}$$

Solve with Hopf-Lax formula:

$$v(x,t) = \min_{\gamma \in \mathbb{R}} \left[\underbrace{t L\left(\frac{x-\gamma}{t}\right) + g(\gamma)}_{= Q(\gamma; x, t)} \right]$$

$$\text{where } L(q) = f^*(q) = \max_{p \in \mathbb{R}} [pq - f(p)] = \frac{q^2}{4}$$

$$Q(\gamma) = \frac{(x-\gamma)^2}{4t} + \gamma \mathbb{1}_{(0,\infty)}^{(\gamma)} + 3\gamma \mathbb{1}_{(-\infty,0)}^{(\gamma)}$$

Critical points:

$$\text{If } \gamma > 0, \text{ then } Q'(\gamma) = \frac{\gamma-x}{2t} + 1 = 0 \Rightarrow \gamma_1 = x - 2t > 0$$

$$\text{If } \gamma < 0, \text{ then } Q'(\gamma) = \frac{\gamma-x}{2t} + 3 = 0 \Rightarrow \gamma_2 = x - 6t < 0$$

At least one of $x > 2t$, $x < 6t$ always hold,
so we can exclude boundary point $\gamma_3 = 0$ as minimizer.

when valid

$$Q(\gamma_1) = x - t, \quad Q(\gamma_2) = 3x - 9t$$

$$v(x, t) = \begin{cases} Q(\gamma_1) & x > 6t \\ \min(Q(\gamma_1), Q(\gamma_2)) & x \in (2t, 6t) \\ Q(\gamma_2) & x < 2t \end{cases}$$

$$= \begin{cases} x - t & x > 4t \\ 3x - 9t & x < 4t \end{cases}$$

$$u(x, t) = v_x(x, t) = \begin{cases} 1 & x > 4t \\ 3 & x < 4t \end{cases}$$

(shock moves along curve $(\underline{x}(t), t) = (4t, t)$).
Rankine-Hugoniot

$$\begin{aligned} \dot{\underline{x}} &= \frac{f(u_l(\underline{x}(t), t)) - f(u_r(\underline{x}(t), t))}{u_l(\underline{x}(t), t) - u_r(\underline{x}(t), t)} \\ &= \frac{9 - 1}{3 - 1} = 4 \quad \text{Ok.} \end{aligned}$$

Problem 4

4a Verify that $u(x,t) = e^{-t} g(x-bt)$ solves

$$\begin{aligned} u_t + b \cdot Du &= -u & x \in \mathbb{R}^d, t \geq 0 \\ u(x,0) &= g(x) & x \in \mathbb{R}^d \end{aligned}$$

where $b \in \mathbb{R}^d$ and $g \in C^1(\mathbb{R}^d)$.

Sol $u(x,0) = e^0 g(x-0) = g(x)$ ok

$$u_x(x,t) = e^{-t} Dg(x-bt)$$

$$\begin{aligned} u_t(x,t) &= -e^{-t} g(x-bt) - e^{-t} b \cdot Dg(x-bt) \\ &= -u - b \cdot \underline{u_x} \end{aligned}$$

4b Use Duhamel's principle to find a solution to

$$\left. \begin{aligned} u_t + b \cdot Du &= -u + f & x \in \mathbb{R}^d, t \geq 0 \\ u(x,0) &= g \end{aligned} \right\} (2)$$

Change of Problem

where $f \in C^1(\mathbb{R}^d \times [0, \infty))$. Verify that what you obtain is a solution.

$$V_t + b \cdot D V = -V + f \quad (3)$$

$$V(x, 0) = 0$$

and let $w(x, t) = e^{-t} g(x - bt)$.

Then if v solves (3),

$u = v + w$ solves (2) as

$$u_t = v_t + w_t = -b \cdot (Dv + Dw) + f - (v + w)$$

$$= -b \cdot Du + f - u \quad \text{ok}$$

and $u(x, 0) = v(x, 0) + w(x, 0) = g(x)$. ok

Remains to solve (3). Let

$$\bar{V}_t(x, t; s) + b \cdot D \bar{V}(x, t; s) = -\bar{V}(x, t; s) \quad t \geq s, x \in \mathbb{R}^d$$

$$\bar{V}(x, s; s) = f(x, s)$$

By 4a) solution

$$\bar{V}(x, t; s) = e^{-(t-s)} f(x - b(t-s), s)$$

Duhamel's principle: $v(x, t) = \int_0^t \bar{V}(x, t; s) ds$

$$= \int_0^t e^{s-t} f(x+b(s-t), s) ds$$

Verification that V solves (3):

$$V(x, 0) = \int_0^0 f(x, s) ds = 0 \text{ ok.}$$

$$V_t = \bar{V}(x, t; t) + \int_0^t \bar{V}_t(x, t; s) ds$$

$$\stackrel{4a}{=} 0 + \int_0^t -b \cdot D\bar{V}(x, t; s) - \bar{V}(x, t; s) ds$$

$$= -b \cdot D \int_0^t \bar{V}(x, t; s) ds - \int_0^t \bar{V}(x, t; s) ds$$

$$= -b \cdot DV(x, t) - \underline{V(x, t)}.$$

write full solution $u(x, t) = V(x, t) + u(x, t)$ out.

Problem 5.

Let $U \subset \mathbb{R}^d$ be open, bounded and connected and let $u \in C^\infty(\mathbb{R}^d)$.

5a Show that if u is harmonic in U , then $D^\alpha u$ is harmonic for any $\alpha \in \mathbb{N}_0^d$.

Sol $\Delta u(D^\alpha u) \stackrel{u \text{ smooth}}{=} D^\alpha(\Delta u) \stackrel{\Delta u=0}{=} 0.$

5b Show that if u_{x_i} is harmonic in U for any $i \in \{1, \dots, d\}$, ← (typo in exam ~~$d=d$~~)

then u satisfies $\Delta u = a$ in U

Sol: Then $\partial_{x_i}(\Delta u) = \Delta(u_{x_i}) = 0 \quad \forall i.$

$$\Rightarrow |\nabla \Delta u(x)|^2 = 0 \quad \forall x \in U \Rightarrow \Delta u = a.$$

5c Assume u satisfies $-\Delta u = f$ in U for a polynomial f of degree $k \in \mathbb{N}$.

Prove the mean-value formula

$$D^\alpha u(x) = \int_{B(x,r)} D^\alpha u(y) dy \quad \text{for any}$$

$\alpha \in \mathbb{N}_0^d$ with $|\alpha| > k$. For what $x \in U$ and $r > 0$ is the formula valid.

(Hint: use mean-val formula for harmonic functions).

Sol For any $|\alpha| > k$, $D^\alpha f = 0$ (as f has degree k).

$$\Rightarrow -\Delta(D^\alpha u) = -D^\alpha(\Delta u) = D^\alpha f = 0.$$

$D^\alpha u$ harmonic + mean-val formula

$$\Rightarrow D^\alpha u(x) = \int_{B(x,r)} D^\alpha u(y) dy$$

$\forall x \in U$ and $r > 0$ s.t. $B(x,r) \subset U$
(i.e. $r < \text{dist}(x, \partial U)$).

Ex Use 5c to prove the following maximum principle for any $\alpha \in \mathbb{N}^d$ s.t. $|\alpha| > k$:

$$D^\alpha u(x) \leq \max_{x \in \partial U} D^\alpha u \quad \forall x \in U.$$

Sol Let $M = \max_U D^\alpha u(x)$ and suppose $D^\alpha u(x) = M$ for some $x \in U$.

Take a $B(x, r) \subset U$. Then

$$M = D^\alpha u(x) \stackrel{5c}{=} \int_{B(x, r)} D^\alpha u(y) dy \leq M$$

$$\Rightarrow D^\alpha u(y) = M \quad \forall y \in B(x, r)$$

$$\text{Let } A = \{y \in U \mid D^\alpha u(y) = M\}.$$

Above arg shows A is open. But

$A = (D^\alpha u)^{-1}(\{M\}) \cap U$ is also closed in U .

U connected $\Rightarrow U$ is any clopen set in $U \Rightarrow A = U$.

Conclusion: $D^\alpha u$ has interior maximizer
 $\Rightarrow D^\alpha u$ is constant in $\bar{U}$

$$\Rightarrow D^\alpha u(x) \leq \max_{\partial U} D^\alpha u$$

Problem 6

Consider

$$\left. \begin{aligned} u_t + f(u)_x &= \varepsilon u_{xx} & x \in (0,1), t \in (0,T] \\ u(0,t) &= u(1,t) = 0 & t \in (0,T] \\ u(x,0) &= g(x) & x \in (0,1) \end{aligned} \right\} (5)$$

with $f(u) = u^3$, $\varepsilon > 0$ is given and

$g \in C([0,1])$ with $g(0) = g(1) = 0$.

Let $u \in C^2((0,1) \times (0,T]) \cap C([0,1] \times [0,T])$ be a solution to (5).

6a Prove that $E(t) = \int_0^1 u^2(x,t) dx$ decreases over time.

Sol Integration by parts yields

$$\begin{aligned} E'(t) &= \int_0^1 2u u_t dx = 2 \int_0^1 \varepsilon u_{xx} u - f(u)_x u dx \\ &= 2\varepsilon [u_x u(\cdot, t)]_{x=0}^1 - 2\varepsilon \int_0^1 (u_x)^2 dx - \int 6u^3 u_x dx \end{aligned}$$

$$= -2\varepsilon \int_0^1 (u_x)^2 dx - \underbrace{\frac{3}{2} \left[u^4(\cdot, t) \right]_{x=0}^1}_{=0} \\ \leq 0$$

using that $u(0, t) = u(1, t) = 0 \quad \forall t$.

6b Prove that (Max principle)

$$\min_{y \in [0,1]} g(y) \leq u(x, t) \leq \max_{y \in [0,1]} g(y).$$

Hint: (Consider $u^\delta(x, t) = u(x, t) - \delta t$ for $\delta > 0$.)

Sol By continuity $g(0) = 0$,

$$\min_{[0,1]} g \leq 0 \leq \max_{[0,1]} g.$$

Suppose $(x, t) \in (0, 1) \times (0, T]$ is a local max of u^δ . Then $u_t^\delta \geq 0$, $u_x^\delta = 0$ and $u_{xx}^\delta \leq 0$ at (x, t)

$$\text{But } u_t^\delta = u_t - \delta = -f(u)_x + \varepsilon u_{xx}^\delta - \delta$$

$$= \varepsilon u_{xx}^{\delta} + 3u^2 \underbrace{u_x^{\delta}}_{=0} - \delta = \varepsilon u_{xx}^{\delta} - \delta \stackrel{\text{contradiction}}{<} 0$$

at $(x, t) \Rightarrow u^{\delta}$ local maxima in $(0, 1) \times (0, T]$

$$\text{so } u^{\delta}(x, t) \leq \max_{[0, 1]} u^{\delta}(x, 0) = \max_{[0, 1]} g$$

limit $\delta \downarrow 0$ gives $u(x, t) \leq \max_{[0, 1]} g$.

Next, consider $u^{\delta}(x, t) := u(x, t) + \delta t$
and $(x, t) \in (0, 1) \times [0, T]$ local min.

Then $u_t^{\delta} \leq 0$, $u_x^{\delta} = 0$, $u_{xx}^{\delta} \geq 0$ at (x, t) .

But $u_t^{\delta} = \varepsilon u_{xx}^{\delta} + \delta \stackrel{\text{contradiction}}{>} 0$ at (x, t)

$\Rightarrow u^{\delta}(x, t) \geq \min_{[0, 1]} g$ and limit $\delta \downarrow 0$

yields $u(x, t) \geq \min_{[0, 1]} g$.

(Ok to answer "a similar procedure yields the lower bound".)

6C Uniqueness. Unlike for the heat eq, we cannot apply the results in 6a or 6b to prove uniqueness for (5).

Why?

sol PDE (5) is not linear, due to $f(u) = u^3$. So if u and v solves (5),

$w = u - v$ does not solve a similar PDE.