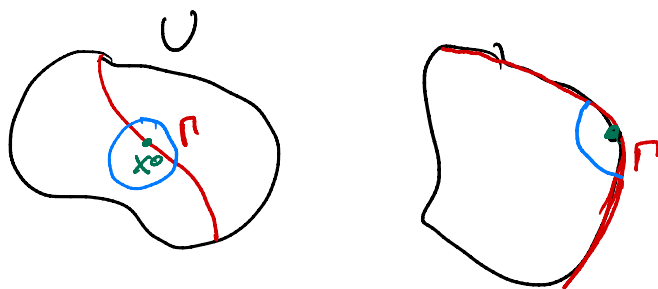


MoC - local existence and uniqueness

Given PDE $F(Du, u, x) = 0$ in $U \subset \mathbb{R}^n$
 $u = g$ on Γ

where $\Gamma \simeq \mathbb{R}^{n-1}$ is part of ∂U or goes through U



Q: When can the characteristics

$$\dot{x} = D_p F(p, z, x)$$

$$\dot{z} = p \cdot D_p F(p, z, x)$$

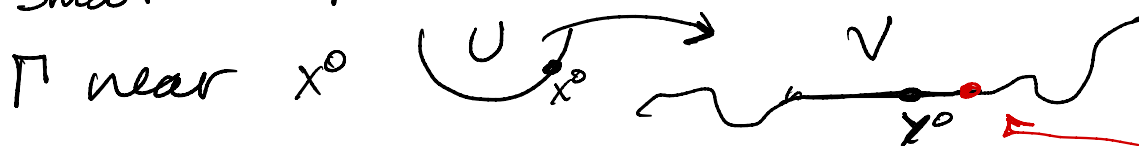
$$\dot{p} = -F_x(p, z, x)p - D_x F(p, z, x)$$

with $x(0) = x^0$, $z(0) = u(x(0)) = g(x^0)$, ...

be used to solve $u(x)$ locally near x^0 ?

Will cover:

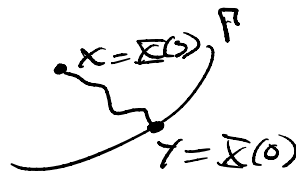
[1] Smooth map $\Phi: U \rightarrow V$ can flatten



with equivalent PDE $G(DV, V, x) = 0$ on V .

[2] Admissible BC/IC $(P(x), z(x), x)$ for $x \in \Gamma$ near x^0 .

[3] Inversion: For each $x \in U$ near $x^0 \in \Gamma$,
 $\exists! \gamma(x) \in \Gamma$ and $s(x) \in \mathbb{R}$ s.t. projected characteristic
 $X(0) = \gamma(x)$ & $X(s(x)) = x$

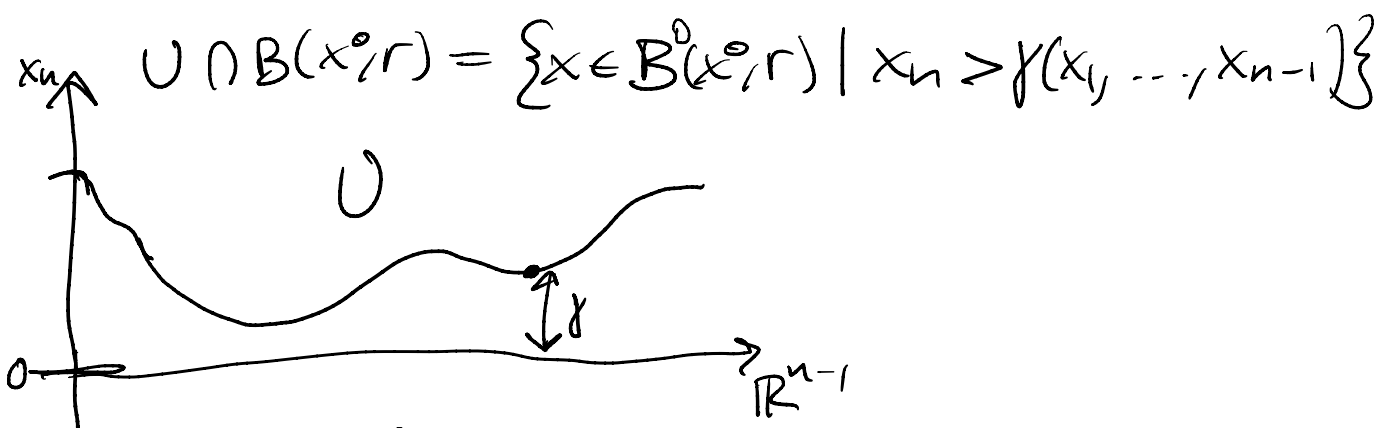


[4] A local solution:

$u(x) := z(\gamma(x), s)$ ($= z(s)$ for with $z(0) = \gamma(x)$)
solves $F(Du, u, x) = 0$ in
neighborhood $N \subset U$ containing $x^0 \in \Gamma$.

1 Local straightening of Γ .

Def For $U \subset \mathbb{R}^n$ open we say ∂U is C^k if for each $x^0 \in \partial U$, $\exists r > 0$ a C^k $f: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ s.t. (upon relabeling coordinates) we have

$$x_n \uparrow \quad U \cap B(x^0, r) = \{x \in B(x^0, r) \mid x_n > f(x_1, \dots, x_{n-1})\}$$


Note $\partial U \cap B(x^0, r) = \{x \in B(x^0, r) \mid x_n = f(x_1, \dots, x_{n-1})\}$.

(If Γ instead goes through U , then Γ is C^k if for $x^0 \in \Gamma$, $\exists r > 0$ $f \in C^k$ s.t.
 $\Gamma \cap B(x^0, r) = \{x \in B(x^0, r) \mid x_n = f(x_1, \dots, x_{n-1})\}.$)

assume $\Gamma \subset \partial U$ and ∂U is C^∞ .

Straightening/flat map

The map $\Phi: U \rightarrow \mathbb{R}^n$ given by

$$\Phi(x) = (x_1, \dots, x_{n-1}, x_n - f(x_1, \dots, x_{n-1})) =: \gamma$$

is then

① C^∞ , as $\gamma \in C^\infty$;

② invertible as

$$\Psi(\gamma) := (\gamma_1, \dots, \gamma_{n-1}, \gamma_n + \gamma(\gamma_1, \dots, \gamma_{n-1})) = x,$$

③ "flattening" near $\gamma^0 = \Phi(x^0)$ in $\Phi(\partial U)$

$$\gamma_n = \psi_n(x) = x_n - \gamma(x_1, \dots, x_{n-1}) \text{ for } x \in \partial U \cap B^0(x^0, r)$$

$$\text{Flat means } \Phi(\partial U \cap B^0(x^0, r)) = \{\gamma \in \Phi(B(x^0, r)) \mid \gamma_n = 0\}$$

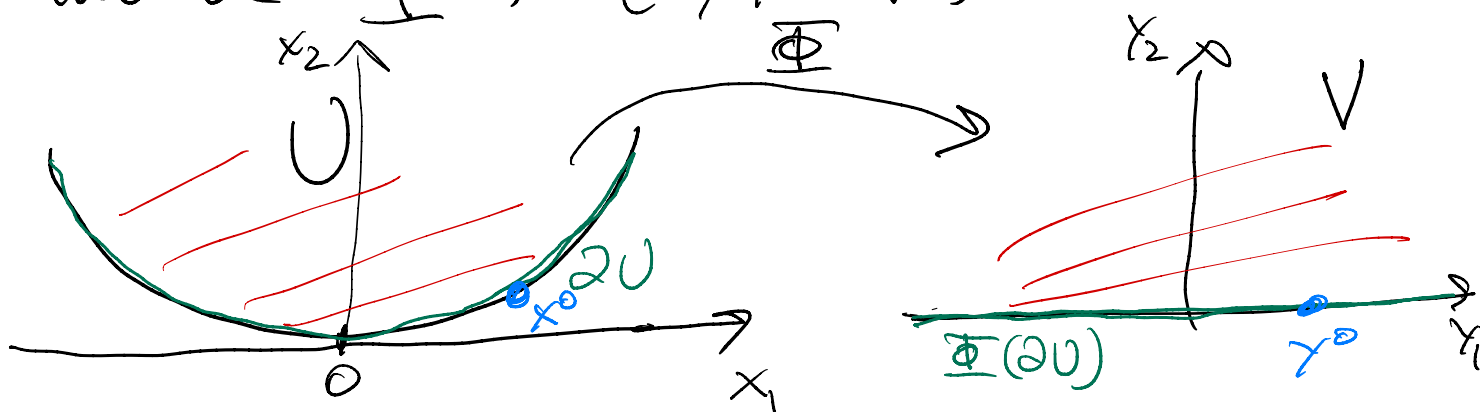
Ex

$$U = \{x \in \mathbb{R}^2 \mid x_2 > x_1^2\} \text{ gives } \partial U = \{x \in \mathbb{R}^2 \mid x_2 = \gamma(x_1) = x_1^2\}$$

$$\text{Flat map: } \gamma = \Phi(x) = (x_1, x_2 - x_1^2)$$

$$V = \Phi(U) = \{\gamma \in \mathbb{R}^2 \mid \gamma_2 > 0\} \text{ with}$$

$$\text{inverse } \Psi(\gamma) = (x_1, \gamma_2 + \gamma_1^2).$$



Consider PDE

$$\left. \begin{array}{l} F(Du, u, x) = 0 \quad \text{in } U \\ u = g \quad \text{on } \Gamma \end{array} \right\} \quad (1)$$

Let u be a smooth solution and

let $\Phi: U \rightarrow V$ be a smooth flat map
near a point $x^0 \in \Gamma$ with a smooth
inverse $\Psi = \Phi^{-1}$. For $x \in U$, let $y = \Phi(x) \in V$.
(so that $x = \Psi(y)$).

Define pushforward $v(y) := u(\Psi(y)) \quad y \in V$.

$$\Rightarrow \quad u(x) = u(\Psi(\Phi(x))) = v(\Phi(x)) \quad x \in U.$$

Then, writing $\Phi(x) = (\Phi^1, \dots, \Phi^n)$,

$$u_{x_i}(x) = \frac{\partial}{\partial x_i} v(\Phi(x)) = \sum_{j=1}^n v_{y_j}(\Phi(x)) \Phi_{x_i}^j(x)$$

$$\Rightarrow Du(x) = \underbrace{Dv(y)}_{\in \mathbb{R}^{1 \times n}} \underbrace{D\Phi(x)}_{\in \mathbb{R}^{n \times n}} = Dv(y) D\Phi(\Psi(y))$$

For $x \in U$, or equivalently $\Phi(x) = y \in V$,

$$\begin{aligned} 0 &\stackrel{(1)}{=} F(Du(x), u(x), x) = F(Dv(y) D\Phi(\Psi(y)), v(y), \Psi(y)) \\ &=: G(Dv(y), v(y), y). \end{aligned}$$

Hence, flat map Φ converts (1) into an equivalent problem

$$\left. \begin{aligned} G(DV(Y), V(Y), Y) &= 0 && \text{in } V \\ V &= g && \text{on } \Phi(\Gamma), \end{aligned} \right\} (2)$$

And near $Y^0 = \Phi(x^0)$, $\Phi(\Gamma)$ is flat:
 $\exists r > 0$ s.t.

$$\Phi(\Gamma) \cap B(Y^0, r) = \{Y \in B(Y^0, r) \mid Y_n = 0\}.$$

For a given $x^0 \in \Gamma$, we may therefore w.l.o.g. assume Γ is flat near x^0 in (1) and write in x -coordinates:

meaning $\Gamma \cap B(x^0, r) = \{x \in B(x^0, r) \mid x_n = 0\}.$

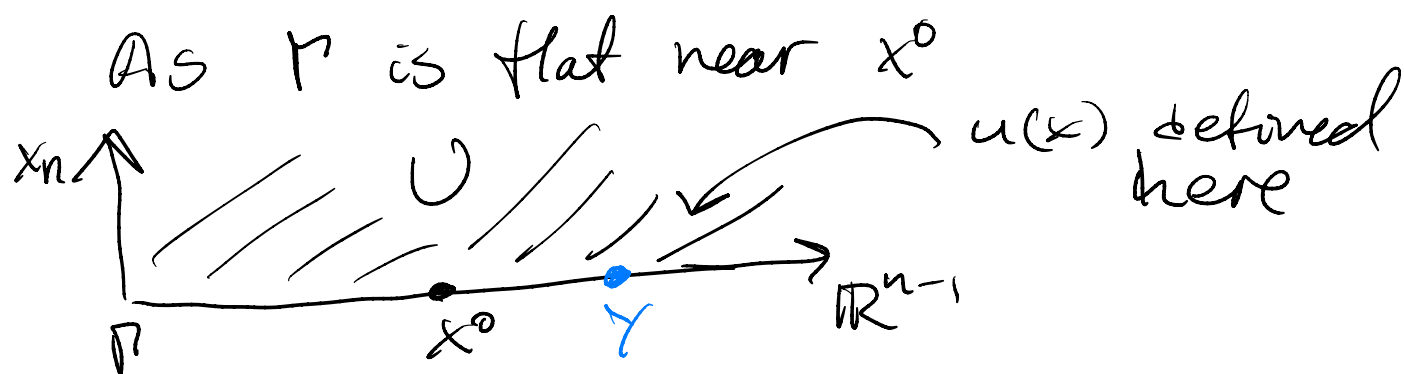
[2] Admissible $I \subset \mathbb{R}$ near $x^0 \in \Gamma$.

Given $X(0) = x^0 = (x_1^0, \dots, x_{n-1}^0, 0)$, we set

- $z(0) = g(x^0)$ (as $z(0) = U(X(0))$)

and as $P_i(0) = U_{x_i}(X(0))$, we set

- $P_i^0 = g_{x_i}(x^0) \quad i = 1, \dots, n-1.$



This means $g(x) = \tilde{g}(x_1, \dots, x_{n-1})$
 near x^0 , so g_{x_n} is not defined / provides
 no information about $u_{x_n}(x^0)$.

Instead, we determine p_n^0 by solving

$$\bullet F(p_1^0, \dots, p_{n-1}^0, p_n^0, z^0, x^0) = 0$$

given $(p_1^0, \dots, p_{n-1}^0), z^0, x^0$.

If a $p_n^0 \in \mathbb{R}$ exists solving ((2) or (1))

(p_n^0, z^0, x^0) is an admissible IC at x^0

(p_n^0 need not exist or be unique.)

Next, for all points $\gamma = (\gamma_1, \dots, \gamma_{n-1}, 0) \in \Gamma$

close to $x^0 \in \Gamma$ we seek to

solve characteristic ODE

$$\dot{X}(s) = D_P F(P, z, X)$$

$$\dot{Z}(s) = D_P F(P, z, X) \cdot P$$

$$\dot{P}(s) = -F_Z(P, z, X)P - D_X F(P, z, X)$$

With the IC

$$X(0) = \gamma, \quad z(0) = g(\gamma) \text{ and}$$

$$P(0) = q(\gamma) \text{ for some } q: \Gamma \rightarrow \mathbb{R}^n$$

$$q_i(\gamma) = q_{x_i}(\gamma) \quad i = 1, \dots, n-1$$

and $q_n(\gamma)$ solving

$$F((q_1(\gamma), \dots, q_{n-1}(\gamma), q_n(\gamma)), g(\gamma), \gamma) = 0 \quad (3)$$

Lemma 3.2.1 Given ∂U smooth,

$u \in C^2(\bar{U})$, $g \in C^2(\partial U)$ and given fixed admissible IC (P^0, z^0, x^0) , there exists a unique solution $q(\gamma)$ for with $q \in C^2(\mathbb{R}^n)$ solving (3) for γ sufficiently close to x^0 , provided $F_{P_n}(P^0, z^0, x^0) \neq 0$.

Proof Recall implicit fcn thm:

If a function $G: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is C^2 ,
 $G(p^0, x^0) = 0$ for some $(p^0, x^0) \in \mathbb{R}^n \times \mathbb{R}^n$
& $D_p G(p^0, x^0) \in \mathbb{R}^{n \times n}$ is invertible,
then $\exists q \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ & $r > 0$ s.t.

$$G(q(\gamma), \gamma) = 0 \quad \forall \gamma \in B(x^0, r)$$

For this case, we consider $G: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$
given by

$$G_i(p, \gamma) := p_i - g_{x_i}(\gamma) \quad i = 1, \dots, n-1$$

$$G_n(p, \gamma) := F(p, g(\gamma), \gamma)$$

$$\Rightarrow G(p^0, x^0) = 0 \quad \text{since}$$

$(p^0, g(x^0), x^0)$ is admissible IC.

And

$$D_p G(p^0, x^0) = \begin{bmatrix} \begin{array}{c|c} \begin{array}{ccc} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{array} & \begin{array}{c} F_{p_1}(p^0, z^0, x^0) \\ \vdots \\ F_{p_{n-1}}(p^0, z^0, x^0) \\ F_{p_n}(p^0, z^0, x^0) \end{array} \end{array} \\ \hline 0 \quad \dots \quad 0 \end{bmatrix}$$

with $\det(D_p G(p^0, x^0)) = F_{p_n}(p^0, z^0, x^0) \neq 0$.



Remark The condition $F_{p_n}(p^0, z^0, x^0) \neq 0$ is called noncharacteristic boundary condition. Motivation: If $\underline{x}(0) = x^0$, then $\nu(\underline{x}(0)) = -e_n$ and $\dot{\underline{x}} = F_p(p^0, z^0, x^0)$

$$\Rightarrow \dot{\underline{x}}(0) \cdot \nu(\underline{x}(0)) = -F_{p_n}(p^0, z^0, x^0) \neq 0,$$

so $\underline{x}(t)$ does not move along Γ near x^0 .

If instead Γ is not flat around x^0 , the condition becomes $D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$.