

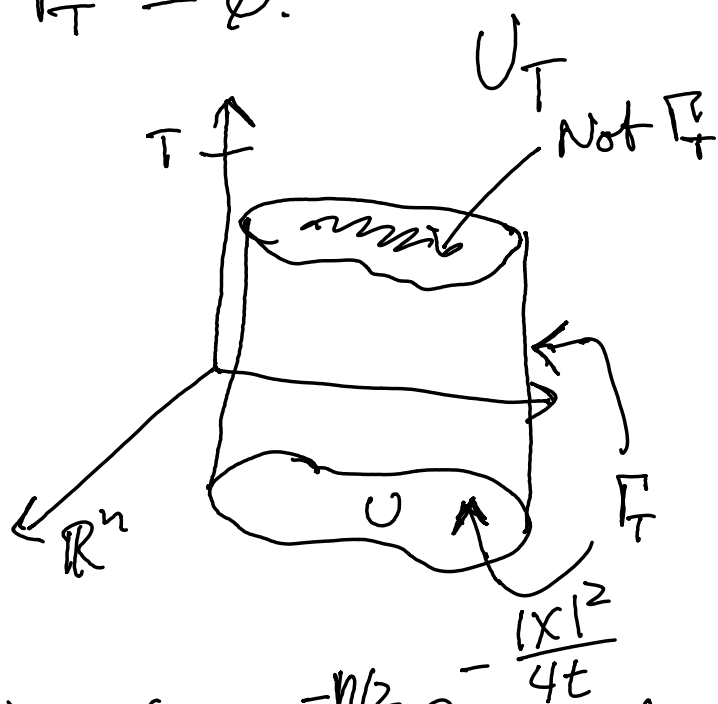
Mean-value formulas

Assume $U \subset \mathbb{R}^n$ is open and bdd. For a fixed $T > 0$, let

$U_T := U \times (0, T]$ denote the parabolic cylinder with "parabolic boundary"

$$\Gamma_T := \overline{U_T} \setminus U_T = (U \times \{0\}) \cup (\partial U \times (0, T))$$

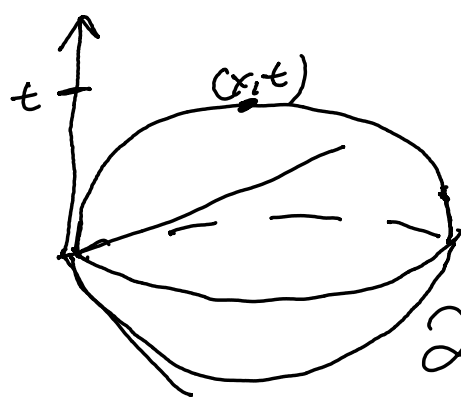
Note: $(\partial U \times \{T\}) \cap \Gamma_T = \emptyset$.



Recall that $\Phi(x, t) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$ $t > 0$
and consider the heat ball

$$E(x, t; r) = \left\{ (y, s) \in \mathbb{R}^{n+1} \mid s < t, \Phi(x-y, t-s) \geq \frac{1}{r^n} \right\}$$

for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $r > 0$.



$E(x, t; r)$ has boundary

$$\partial E = \{(y, s) \mid s < t, \Phi(x-y, t-s) = r^{-n}\} \cup \{(x, t)\}.$$

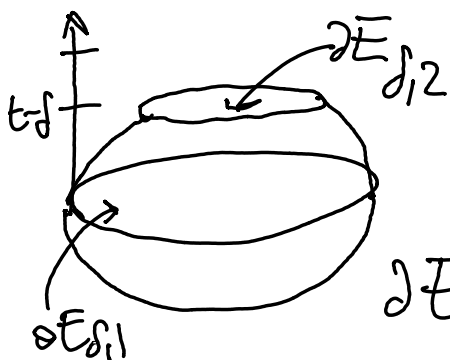
Note that for fixed $s < t$,

$$(y, s) \in E(x, t; r) \Leftrightarrow e^{-\frac{|x-y|^2}{4(t-s)}} \leq \frac{r^n}{(4\pi(t-s))^{n/2}}$$

$$\Leftrightarrow \frac{|x-y|^2}{4(t-s)} \leq n \log(r) - \frac{n}{2} \log(4\pi(t-s)) \quad (4)$$

For $\delta < t$, sufficiently small, consider closed set

$$E_\delta(x, t; r) = E(x, t; r) \cap \{(y, s) \mid s \leq t - \delta\}$$



$$\partial E_\delta = \overbrace{(\partial E \cap \{(y, s) \mid s \leq t - \delta\})}^{\partial E_{\delta,1}} \cup \partial E_{\delta,2}$$

$$\partial E_{\delta,2} \stackrel{(4)}{=} B(x, 2\sqrt{(n \log(r) - \frac{n}{2} \log(4\pi\delta))\delta})$$

Note $|\partial E_{\delta,2}| \leq C (\log(\delta) \delta)^{\frac{n}{2}}$ for $\delta < \frac{1}{4\pi}$.

$$\text{and } \frac{|x-y|^2}{4(t-s)} \leq \underbrace{C + \log(4\pi(t-s))}_{=: h(s)}, \text{ with}$$

$h \in C(E(x,t;r) \setminus \{x,t\})$ and integrable on $E(x,t;r)$.

Theorem 2.3.3

Let $u \in C_1^2(U_T)$ solve the heat equation

$u_t = \Delta u$ in U_T . Then

$$u(x,t) = \frac{1}{4r^n} \int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dy ds$$

for any $E(x,t;r) \subset U_T$.

Note • $\frac{1}{4r^n} \int_{E(x,t;r)} \frac{|x-y|^2}{(t-s)^2} dy ds = 1$, so above

is a weighted average of u .

• $u(x,t)$ only depends on values $u(y,s)$ from the past.

Proof As $E(x, t; r)$ is compact, let $\bar{V} \subset U$ be compact and let $\bar{V} \times [a, b] \supset E$ for $[a, b] \subset (0, T]$. Define

$$u^\varepsilon(t, x) = \int_{U_{T, \varepsilon}} \eta_\varepsilon(x - \gamma) u(t, \gamma) d\gamma.$$

Then $D^\alpha u \in C(\bar{V} \times [a, b])$ for any $|\alpha| \leq 1 \Rightarrow D^\alpha u^\varepsilon(t, x) \xrightarrow{\varepsilon \rightarrow 0} D^\alpha u(t, x)$ uniformly in $\bar{V} \times [a, b]$.

Further, $D_x^\alpha u^\varepsilon, D_t^\alpha u^\varepsilon \in C(\bar{V} \times [a, b]) \quad \forall \alpha \in \mathbb{N}_0^n$,

and $u_t^\varepsilon(t, x) - \Delta u^\varepsilon(t, x) = 0$ in $\bar{V} \times [a, b]$.

Consider

$$\phi(r) := \frac{1}{r^n} \int_{E(x, t; r)} u(\gamma, s) \frac{|x - \gamma|^2}{(t - s)^2} d\gamma ds$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{1}{r^n} \int_{E(0, 0; r)} \underbrace{u^\varepsilon(x - \gamma, t - s)}_{=: v^\varepsilon(\gamma, s)} \frac{|\gamma|}{s^2} d\gamma ds$$

$$= \lim_{s, \varepsilon \rightarrow 0} \frac{1}{r^n} \int_{E_{r/2}(0, 0; r)} v^\varepsilon(\gamma, s) \frac{|\gamma|}{s^2} d\gamma ds \quad \leftarrow \text{cut off ball}$$

Fix $\delta, \varepsilon > 0$ small, let $E_{r\delta}(r) := E_{r\delta}(0, 0; r)$

Then

$$\Phi_{\delta, \varepsilon}(r) = \frac{1}{r^n} \int_{E_{r\delta}(r)} V^\varepsilon(\gamma, s) \frac{|\gamma|^2}{s^2} d\gamma ds \quad \left[\overline{\gamma} = \frac{\gamma}{r}, \overline{s} = \frac{s}{r^2} \right]$$

$$= \int_{E_\delta(1)} \underbrace{V^\varepsilon(r\gamma, r^2s)}_{\tilde{V}(\gamma, s)} \frac{|\gamma|^2}{s^2} d\gamma ds$$

$$\rightarrow \Phi(r) \quad \text{as } \delta, \varepsilon \rightarrow 0.$$

And as $D_Y^\alpha \tilde{h}, D_Y^\alpha \tilde{h}_s \in C^\infty(E_\delta(1))$, $\Phi_{\delta, \varepsilon}'(r) \rightarrow \Phi'(r)$ and

$$\Phi_{\delta, \varepsilon}'(r) = \int_{E_\delta(1)} \sum_{i=1}^n V_{\gamma_i}^\varepsilon(r\gamma, r^2s) \gamma_i \frac{|\gamma|^2}{s^2} + 2r V_s^\varepsilon(r\gamma, r^2s) \frac{|\gamma|^2}{s} d\gamma ds$$

$$= \frac{1}{r^{n+1}} \int_{E_{r\delta}(r)} \left(D_Y V^\varepsilon(\gamma, s) \cdot \gamma \right) \frac{|\gamma|^2}{s^2} + 2 V_s^\varepsilon(\gamma, s) \frac{|\gamma|^2}{s} d\gamma ds$$

$$=: A + B$$

Let $\psi(\gamma, s) \stackrel{(4)}{=} -\frac{n}{2} \log(-4\pi s) + \frac{|\gamma|^2}{4s} + n \log r$ and

use shorthand $\tilde{E}_\delta(r) = E_{r\delta}(r)$. $\psi = 0$ on $\partial \tilde{E}_{\delta, 1}(r)$

and $|\psi(\gamma, s)| \stackrel{(4)}{\leq} \frac{n}{2} \log(4\pi s) + n \log(r)$ on $\partial \tilde{E}_s(r)$.

so that $\int_{\partial \tilde{E}_s(r)} |\psi(\gamma, s)| dS(\gamma, s) \leq c |\partial E_{r/2}(r)| |\log(s)|$
 $= O(s^{n/2} |\log(s)|) = O(s^{1/3})$

and $4 \sum_{i=1}^n \gamma_i \psi_{\gamma_i} = 2 \frac{|\gamma|^2}{s}$ yields

$$B = \frac{1}{r^{n+1}} \int_{\tilde{E}_s(r)} 4 V_s^\varepsilon \sum_{i=1}^n \gamma_i \psi_{\gamma_i} d\gamma ds$$

$$= O(s^{1/3}) - \underbrace{\frac{1}{r^{n+1}} \int_{\tilde{E}_s(r)} 4 \sum_{i=1}^n V_{s\gamma_i}^\varepsilon \gamma_i \psi d\gamma ds}_{C} - \frac{1}{r^{n+1}} \int_{\tilde{E}_s(r)} 4n V_s^\varepsilon \psi d\gamma ds$$

$$C = \frac{1}{r^{n+1}} \int_{\tilde{E}_s(r)} 4 \sum_{i=1}^n V_{s\gamma_i}^\varepsilon \gamma_i \psi d\gamma ds + O(s^{1/3})$$

$$= \frac{1}{r^{n+1}} \int_{\tilde{E}_s(r)} (D_\gamma V^\varepsilon \cdot \gamma) \left(\frac{2n}{s} - \frac{|\gamma|^2}{s} \right) d\gamma ds + O(s^{1/3})$$

$$= -A + \frac{1}{r^{n+1}} \int_{\tilde{E}_s(r)} (D_\gamma V^\varepsilon \cdot \gamma) \frac{2n}{s} d\gamma ds + O(s^{1/3})$$

So $\phi'_{s, \varepsilon}(r) = A + B = \frac{1}{r^{n+1}} \int_{\tilde{E}_s(r)} (D_\gamma V^\varepsilon \cdot \gamma) \frac{2n}{s} d\gamma ds - \underbrace{4n V_s^\varepsilon \psi}_{\Delta V^\varepsilon} d\gamma ds + O(s^{1/3})$

$$= \frac{1}{r^{n+1}} \int_{E_\delta(r)} \left(D_\gamma V^\varepsilon \cdot \gamma \right) \frac{2n}{s} - 4n D_\gamma V^\varepsilon \cdot \underbrace{D_\gamma V}_{= \frac{\gamma}{2s}} d\gamma ds + O(\delta^{1/3})$$

$$= O(\delta^{1/3})$$

$$\Rightarrow \phi'(r) = \lim_{\varepsilon/\delta \rightarrow 0} \phi'_{\varepsilon/\delta}(r) = 0$$

$$\begin{aligned} \text{so } \phi(r) &= \lim_{t \rightarrow 0} \phi(t) = \lim_{t \rightarrow 0} \frac{1}{4} \int_{E(1)} V(t\gamma, t^2s) \frac{|\gamma|^2}{s^2} d\gamma ds \\ &= V(0,0) \underbrace{\frac{1}{4} \int_{E(1)} \frac{|\gamma|^2}{s^2} d\gamma ds}_{=1} \quad (\text{Exercise}) \\ &= u(x,t) \end{aligned}$$

$$\text{as } V(\gamma, s) := u(x-\gamma, t-s).$$

