

MAT4301 – Suggested exercises – week 36

September 3, 2025

1. Compute the gradient Du of the following functions:

(a) $u(x) = \sin(x_1 x_2^2 - x_3^3)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

(b) $u(x) = |x|$ for $x \in \mathbb{R}^n$

(c) $u(x) = |x|^2$ for $x \in \mathbb{R}^n$

(Here and elsewhere, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ denotes the Euclidean norm of x .)

2. Write out the expression

$$\sum_{|\alpha|=2} \alpha! x^\alpha$$

where $x = (x_1, x_2)$ is some point in $\mathbb{R}^2$.

(Here and elsewhere we use the convention that α denotes a multiindex, so the sum runs over all pairs of nonnegative integers $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ whose sum $|\alpha| = \alpha_1 + \alpha_2$ equals 2 and the multiindex factorial is defined as $\alpha! = \alpha_1! \alpha_2!$.)

3. Compute the partial derivative $D^\alpha u$ for all multiindices α of length $|\alpha| = 1$ and $|\alpha| = 2$, for the functions $u(x) = |x|^2$ and $u(x) = |x|$ (where $x \in \mathbb{R}^n$).
4. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a given function, fix a point $x \in \mathbb{R}^2$ and define $g(t) = f(tx)$. Write out $g'(t)$ and $g''(t)$ in terms of partial derivatives of f . Use multiindex notation.
5. Problems 1.5.3, 1.5.4, 1.5.5 in Evans.

Hint for 1.5.3: Use induction on n , not k . Recall the binomial theorem, $(a+b)^m = \sum_{r=0}^m \binom{m}{r} a^r b^{m-r}$, where $\binom{m}{r} = \frac{m!}{r!(m-r)!}$ are the binomial coefficients.

Hint for 1.5.4: Use induction on n . Recall Leibniz' formula in one dimension: $\partial_{x_n}^k (fg) = \sum_{r=0}^k \binom{k}{r} \partial_{x_n}^r f \partial_{x_n}^{k-r} g$ for functions $f, g \in C^k(\mathbb{R}^n)$.

Hint for 1.5.5: As mentioned in the exercise, define $g(t) = f(tx)$ for $t \in \mathbb{R}$. Write down the k th order Taylor expansion for $g(1)$ (including error term), expanded around $t = 0$. Show that the m th derivative of g can be written as $g^{(m)}(t) = \sum_{|\alpha|=m} \binom{m}{\alpha} x^\alpha D^\alpha f(tx)$. To this end:

- Show first that

$$g^{(m)}(t) = \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n x_{i_1} \cdots x_{i_m} \partial_{x_{i_1}} \cdots \partial_{x_{i_m}} f(tx).$$

- Next, recall the fact that for a multiindex α of length $|\alpha| = m$, the number $\binom{m}{\alpha} = \frac{m!}{\alpha_1! \cdots \alpha_n!}$ is the number of ways to extract m balls of n different colors from a bag, picking α_1 of the first color, α_2 of the second color, and so on. Use this fact to rewrite the above expression for $g^{(m)}(t)$ in multiindex notation.

6. Problem 2.5.1

Hint: Try to extend the example in Section 2.1.2 to the setting with f replaced by cu , and solve the resulting ODE for $\frac{d}{dt}u(x(t), t)$ along characteristics.

7. Problem 2.5.2

Hint: Apply the chain rule when differentiating $\Delta u(Ox)$ and try to utilize that $O^T O = I$.

Solution: (Many were kindly provided by Ulrik Skre Fjordholm.)

1. (a) We have $u_{x_1}(x) = x_2^2 \cos(x_1 x_2^2 - x_3)$, $u_{x_2}(x) = 2x_1 x_2 \cos(x_1 x_2^2 - x_3)$ and $u_{x_3}(x) = -\cos(x_1 x_2^2 - x_3)$, so

$$Du(x) = \cos(x_1 x_2^2 - x_3) \begin{pmatrix} x_2^2 \\ 2x_1 x_2 \\ -1 \end{pmatrix}.$$

- (b) We have $u_{x_i}(x) = 2x_i$, so $Du(x) = 2x$.

- (c) We have $u_{x_i}(x) = \frac{1}{2\sqrt{x_1^2 + \cdots + x_n^2}} 2x_i = \frac{x_i}{|x|}$, so $Du(x) = \frac{x}{|x|}$.

2. The two-dimensional multiindices of length $|\alpha| = 2$ are $(2, 0)$, $(1, 1)$ and $(0, 2)$. Therefore,

$$\sum_{|\alpha|=2} \alpha! x^\alpha = (2, 0)! x^{(2,0)} + (1, 1)! x^{(1,1)} + (0, 2)! x^{(0,2)} = 2x_1^2 + x_1 x_2 + 2x_2^2.$$

3. The partial derivatives D^α when $|\alpha| = 1$ are the components of the gradient of the function, which we already computed.

$u(x) = |x|^2$ We have $u_{x_i}(x) = 2x_i$. If $j \neq i$ then $u_{x_i x_j}(x) = 0$, while $u_{x_i x_i}(x) = 2$. Hence, we can write $D^\alpha u(x) = 2\delta_{\alpha_1, \alpha_2}$, where δ is the Kronecker delta function:

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

$u(x) = |x|$ We have $u_{x_i}(x) = \frac{x_i}{|x|}$. We have $\left(\frac{1}{|x|}\right)_{x_j} = -\frac{1}{|x|^2} |x|_{x_j} = -\frac{x_j}{|x|^3}$.

Hence, if $j \neq i$ then $u_{x_i x_j}(x) = -\frac{x_i x_j}{|x|^3}$. On the other hand, for $i = j$ we

get $u_{x_i x_i}(x) = \frac{1}{|x|} - \frac{x_i^2}{|x|^3}$. Summarizing, we have for any α of length $|\alpha| = 2$

$$D^\alpha u(x) = \frac{\delta_{\alpha_1, \alpha_2}}{|x|} - \frac{x_{\alpha_1} x_{\alpha_2}}{|x|^3}.$$

4. We have

$$g'(t) = f_{x_1}(tx)x_1 + f_{x_2}(tx)x_2 = x_1 D^{(1,0)} f(tx) + x_2 D^{(0,1)} f(tx)$$

and

$$\begin{aligned} g''(t) &= x_1^2 f_{x_1 x_1}(tx) + 2x_1 x_2 f_{x_1 x_2}(tx) + x_2^2 f_{x_2 x_2}(tx) \\ &= x_1^2 D^{(2,0)} f(tx) + 2x_1 x_2 D^{(1,1)} f(tx) + x_2^2 D^{(0,2)} f(tx). \end{aligned}$$

5. Follow the hints.

6. Along the characteristic $x(t) = bt$, one obtains for $z(t) = u(x(t), t)$ that $\dot{z} = -cz(t)$. Solving this ODE, $z(t) = e^{-ct} z(0)$, which yields the solution

$$u(y, t) = e^{-ct} g(y - bt).$$

7. Let $y = Ox$, note that $y_j = \sum_k o_{jk} x_k$ and $\partial y_j / \partial x_i = o_{ji}$. By the chain rule,

$$v_{x_i}(x) = \sum_j u_{y_j}(y) o_{ji}$$

and

$$\partial_{x_i x_i} v(x) = \sum_{j,k=1}^n u_{y_j y_k}(y) o_{ki} o_{ji}$$

Hence

$$\Delta v(x) = \sum_{j,k=1}^n u_{y_j y_k}(y) \underbrace{\sum_{i=1}^n o_{ki} o_{ji}}_{\delta_{kj}} = \sum_{j=1}^n u_{y_j y_j}(y) = \Delta_y u(y) = 0,$$

where δ_{kj} denotes the Kronecker delta.