

MAT4301 – Suggested exercises – week 37

September 4, 2025

1. Compute the integral

$$\int_{B(0,1)} \operatorname{div} f(x) \, dx$$

where $B(0, 1)$ is the unit ball in $\mathbb{R}^3$ and $f(x) = |x|^2 x$. What do you get when $B(0, 1)$ is the unit ball (or *disc*) in $\mathbb{R}^2$?

Hint: Use the divergence theorem (Theorem 1(ii) in §C.2).

2. Use the Gauss–Green theorem (Theorem 1(i) in §C.2) to prove all of the other identities in §C.2.

3. A function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is *locally integrable* if for every bounded set $K \subset \mathbb{R}^n$, the integral

$$\int_K |u(x)| \, dx$$

is finite.

- (a) Show that $u(x) = \log |x|$ for $x \in \mathbb{R}^n$ is locally integrable, for any number of dimensions $n \in \mathbb{N}$.

- (b) Let $u(x) = |x|^p$ for $x \in \mathbb{R}^n$ and let $p \in \mathbb{R}$ be a given number. For what values of p is this function locally integrable?

Hint: If u is bounded on K (i.e., $\exists C > 0$ such that $|u(x)| \leq C$ for all $x \in K$) then u is integrable over K , so it suffices to concentrate on bounded domains K where u is unbounded.

4. Consider the Poisson equation

$$-\Delta u = f \quad (x \in \mathbb{R}^n). \quad (1)$$

- (a) Write out this equation in $n = 1$ dimensions (the boundary-value problem ODE).

- (b) Find all solutions of (1) in $n = 1$ dimensions when

$$f \equiv 0 \quad \text{and} \quad f(x) = e^{\lambda x} \text{ for some } \lambda \in \mathbb{R}.$$

- (c) Solve the following *boundary value problem* on the interval $(a, b) \subset \mathbb{R}$,

$$\begin{cases} -u''(x) = 1 & (x \in (a, b)) \\ u(a) = -1, u(b) = 2 \end{cases}$$

5. Solve the *initial value problem*

$$\begin{cases} u_t + b \cdot Du = 0 & (x \in \mathbb{R}^2, t > 0) \\ u(x, 0) = g(x) & (x \in \mathbb{R}^2) \end{cases} \quad (2)$$

where $u: \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}$ is the unknown, $b = (1, 3)$ and $g(x) = \sin(x_1) \cos(x_2)$.

6. Let $U \subset \mathbb{R}^n$ be open and show that if $u \in C(U)$ satisfies

$$u(x) = \oint_{B(x,r)} u(y) dy \quad \text{for each } B(x,r) \subset U,$$

then

$$u(x) = \oint_{\partial B(x,r)} u(y) dS(y) \quad \text{for each } B(x,r) \subset U.$$

7. Problems 2.5.4 and 2.5.5 in Evans