

MAT4301 – Suggested exercises – week 37

September 14, 2025

1. Compute the integral

$$\int_{B(0,1)} \operatorname{div} f(x) \, dx$$

where $B(0, 1)$ is the unit ball in $\mathbb{R}^3$ and $f(x) = |x|^2 x$. What do you get when $B(0, 1)$ is the unit ball (or *disc*) in $\mathbb{R}^2$?

Hint: Use the divergence theorem (Theorem 1(ii) in §C.2).

2. Use the Gauss–Green theorem (Theorem 1(i) in §C.2) to prove all of the other identities in §C.2.

3. A function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is *locally integrable* if for every bounded set $K \subset \mathbb{R}^n$, the integral

$$\int_K |u(x)| \, dx$$

is finite.

- (a) Show that $u(x) = \log |x|$ for $x \in \mathbb{R}^n$ is locally integrable, for any number of dimensions $n \in \mathbb{N}$.

- (b) Let $u(x) = |x|^p$ for $x \in \mathbb{R}^n$ and let $p \in \mathbb{R}$ be a given number. For what values of p is this function locally integrable?

Hint: If u is bounded on K (i.e., $\exists C > 0$ such that $|u(x)| \leq C$ for all $x \in K$) then u is integrable over K , so it suffices to concentrate on bounded domains K where u is unbounded.

4. Consider the Poisson equation

$$-\Delta u = f \quad (x \in \mathbb{R}^n). \quad (1)$$

- (a) Write out this equation in $n = 1$ dimensions (the boundary-value problem ODE).

- (b) Find all solutions of (1) in $n = 1$ dimensions when

$$f \equiv 0 \quad \text{and} \quad f(x) = e^{\lambda x} \text{ for some } \lambda \in \mathbb{R}.$$

- (c) Solve the following *boundary value problem* on the interval $(a, b) \subset \mathbb{R}$,

$$\begin{cases} -u''(x) = 1 & (x \in (a, b)) \\ u(a) = -1, u(b) = 2 \end{cases}$$

5. Solve the *initial value problem*

$$\begin{cases} u_t + b \cdot Du = 0 & (x \in \mathbb{R}^2, t > 0) \\ u(x, 0) = g(x) & (x \in \mathbb{R}^2) \end{cases} \quad (2)$$

where $u: \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}$ is the unknown, $b = (1, 3)$ and $g(x) = \sin(x_1) \cos(x_2)$.

6. Let $U \subset \mathbb{R}^n$ be open and show that if $u \in C(U)$ satisfies

$$u(x) = \oint_{B(x,r)} u(y) dy \quad \text{for each } B(x,r) \subset U,$$

then

$$u(x) = \oint_{\partial B(x,r)} u(y) dS(y) \quad \text{for each } B(x,r) \subset U.$$

7. Problems 2.5.4 and 2.5.5 in Evans

Solution: Many solutions were kindly provided by Ulrik Skre Fjordholm.

1. We use the divergence theorem:

$$I := \int_{B(0,1)} \operatorname{div} f(x) dx = \int_{\partial B(0,1)} f(x) \cdot \nu(x) dS(x) = \int_{\partial B(0,1)} |x|^2 x \cdot x dS(x) = |\partial B(0,1)|$$

where we have used the fact that $\nu(x) = x$ on the unit sphere. We can look up the area $|\partial B(0,1)|$ and find the answer $I = 4\pi$.

In $\mathbb{R}^2$ we get instead $I = |\partial B(0,1)| = 2\pi$, the length of the unit circle.

2. For $u: U \rightarrow \mathbb{R}^n$ we assume that the identity $\int_U u_{x_i} dx = \int_{\partial U} u \nu^i dS$ holds.

Divergence theorem:

$$\int_U \operatorname{div} u dx = \sum_{i=1}^n \int_U u_{x_i} dx = \sum_{i=1}^n \int_{\partial U} u^i \nu^i dS(x) = \int_{\partial U} u \cdot \nu dS.$$

Integration by parts:

$$\int_U u_{x_i} v dx = \int_U (uv)_{x_i} - uv_{x_i} dx = \int_{\partial U} uv \nu^i dS - \int_U uv_{x_i} dx,$$

where we in the first step have used the product rule $u_{x_i} v = (uv)_{x_i} - uv_{x_i}$.

Theorem 3(i):

$$\int_U \Delta u dx = \sum_{i=1}^n \int_U (u_{x_i})_{x_i} dx = \sum_{i=1}^n \int_{\partial U} u_{x_i} \nu^i dS = \int_{\partial U} Du \cdot \nu dS.$$

Theorem 3(ii):

$$\begin{aligned}
\int_U Du \cdot Dv \, dx &= \sum_{i=1}^n \int_U u_{x_i} v_{x_i} \, dx = \sum_{i=1}^n \int_U (u_{x_i} v)_{x_i} - u_{x_i x_i} v \, dx \\
&= \sum_{i=1}^n \int_{\partial U} u_{x_i} v v^i \, dS - \sum_{i=1}^n \int_U u_{x_i x_i} v \, dx \\
&= \int_{\partial U} \frac{\partial u}{\partial \nu} v \, dS - \int_U v \Delta u \, dx
\end{aligned}$$

where we denote $\frac{\partial u}{\partial \nu} = \nu \cdot Du$.

Theorem 3(iii):

$$\begin{aligned}
\int_U u \Delta v - v \Delta u \, dx &= \sum_{i=1}^n \int_U u v_{x_i x_i} - v u_{x_i x_i} \, dx \\
&= \sum_{i=1}^n \int_U (u v_{x_i})_{x_i} - u_{x_i} v_{x_i} - (v u_{x_i})_{x_i} + v_{x_i} u_{x_i} \, dx \\
&= \sum_{i=1}^n \int_{\partial U} u v_{x_i} v^i - v u_{x_i} v^i \, dS \\
&= \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \, dS.
\end{aligned}$$

3. (a) We only need to check that u is integrable near its singularity at $x = 0$. Let, say, $K = B(0, 1)$; then

$$\begin{aligned}
\int_K |\log |x|| \, dx &= \int_0^1 \int_{\partial B(0,r)} -\log r \, dS(x) \, dr = - \int_0^1 |\partial B(0, r)| \log r \, dr \\
&= -|\partial B(0, 1)| \int_0^1 r^{n-1} \log r \, dr,
\end{aligned}$$

which is finite for any $n \geq 1$. Hence, u is locally integrable. Here, we have first converted to polar coordinates and then used the fact that the (hyper-)area $|\partial B(0, r)|$ of the (hyper-)sphere $\partial B(0, r)$ equals $r^{n-1} |\partial B(0, 1)|$.

- (b) If $p \geq 0$ then the function is locally bounded (i.e., bounded on every bounded set), so it's locally bounded. If $p < 0$ then u is bounded away from $x = 0$, so we only need to check integrability near 0. If, say $K = B(0, 1)$ then

$$\begin{aligned}
\int_K u(x) \, dx &= \int_0^1 \int_{\partial B(0,r)} r^p \, dS(x) \, dr = \int_0^1 r^p |\partial B(0, r)| \, dr \\
&= |\partial B(0, 1)| \int_0^1 r^p r^{n-1} \, dr = |\partial B(0, 1)| \int_0^1 r^{p+n-1} \, dr \\
&= |\partial B(0, 1)| \frac{1}{p+n},
\end{aligned}$$

where the last step is only true if $p + n - 1 > -1$, that is, $p > -n$; otherwise, the integral is infinite.

To conclude, $u(x) = |x|^p$ is locally integrable if and only if $p > -n$.

4. (a) The equation in $n = 1$ dimensions states

$$-u''(x) = f(x) \quad (x \in \mathbb{R}).$$

- (b) If $f \equiv 0$ then $u'' \equiv 0$, that is, $u(x) = Ax + B$ for constants $A, B \in \mathbb{R}$.

If $f(x) = \exp(\lambda x)$ then integrating twice yields

$$\begin{aligned} u(x) &= Ax + B - \int^x \int^y \exp(\lambda z) dz dy \\ &= \begin{cases} Ax + B - x^2/2 & \lambda = 0 \\ Ax + B - \exp(\lambda x)/\lambda^2 & \lambda \neq 0. \end{cases} \end{aligned}$$

- (c) Integrating the equation $u''(x) = -1$ twice yields $u(x) = Ax + B - \frac{x^2}{2}$ for constants $A, B \in \mathbb{R}$. Inserting the boundary data yields

$$-1 = u(a) = Aa + B - \frac{a^2}{2}, \quad 2 = Ab + B - \frac{b^2}{2}.$$

Solving for A and B yields

$$A = \frac{6 + b^2 - a^2}{2(b - a)}, \quad B = \frac{a^2}{2} - aA - 1.$$

5. The method of characteristics shows that $s \mapsto u(x + sb, t + s)$ is constant, for any choice of x and t . Setting $s = 0$ and $s = -t$ then yields

$$u(x, t) = u(x - tb, 0) = g(x - tb) = \sin(x_1 - t) \cos(x_2 - 3t).$$

6. For each $B(x, r) \subset U$, the mean-value formula and integration in polar coordinates yield that

$$\alpha(n)r^n u(x) = \int_{B(x, r)} u(y) dy = \int_0^r \int_{\partial B(x, r)} u(y) dS(y) dr$$

Taking the derivative d/dr yields

$$n\alpha(n)r^{n-1}u(x) = \int_{\partial B(x, r)} u(y) dS(y) \implies u(x) = \oint_{\partial B(x, r)} u(y) dS(y),$$

where the last step follows from $\text{Vol}(\partial B(x, r)) = n\alpha(n)r^{n-1}$.

7. Problem 2.5.4: Suppose $y \in U$ is a maximizer of the harmonic function $u^\varepsilon(x) = u(x) + \varepsilon|x|^2$. Then in a neighborhood around y , the single-variable function

$$f_i(z) = u^\varepsilon(y_1, \dots, y_{i-1}, z, y_{i+1}, \dots, y_n)$$

must have a maximizer at $z = y_i$, requiring that $f_i''(y_i) = u_{x_i x_i}^\varepsilon(y) = u_{x_i x_i}(x) + 2\varepsilon \leq 0$. But by the harmonicity of u , we have that

$$\sum_{i=1}^n f_i''(y_i) = \Delta u(y) + 2n\varepsilon = 2n\varepsilon > 0,$$

contradicting that all summands are non-positive. We conclude that u^ε has no interior maximizer, so that

$$\sup_U u^\varepsilon \leq \max_{\partial U} u^\varepsilon.$$

Taking the limit $\varepsilon \rightarrow 0$ and using that $\max_{\overline{U}} \varepsilon |x|^2 \rightarrow 0$, since U is bounded, yields

$$\sup_U u \leq \max_{\partial U} u.$$

8. Problem 2.5.5:

- (a) By similar steps as when deriving the mean-value formula, we obtain for the function

$$\varphi(r) := \int_{\partial B(x,r)} u(y) dS(y) = \int_{\partial B(0,1)} u(x + ry) dS(y),$$

using the divergence theorem, that

$$\varphi'(r) = \int_{\partial B(0,1)} Du(x + ry) \cdot y dS(y) = \frac{r}{n} \int_{B(x,r)} \Delta u(z) dz \geq 0.$$

Since $u(x) = \lim_{r \rightarrow 0} \varphi(r)$, this implies that $\varphi(r) \geq u(x)$ for all valid $r \geq 0$ and

$$\int_{B(x,r)} u(y) dy = \frac{1}{\alpha(n)r^n} \int_0^r n\alpha(n)s^{n-1} \underbrace{\varphi(s)}_{\geq u(x)} ds \geq u(x).$$

- (b) Suppose $x \in U$ is a maximizer of u and that $B(x, r) \subset U$, and write $M = u(x)$. Then

$$M = u(x) \leq \int_{B(x,r)} \underbrace{u(y)}_{\leq M} dy \leq M$$

implies that $u = M$ on the whole ball. Consider the set of maximizing points in U , defined by $A = \{x \in U \mid u(x) = M\}$. Every $x \in A$ is an interior point (as u takes the value M a neighborhood $B^o(x, r)$ around it), so A is an open set. But $A = u^{-1}(M) \cap U$ is also a closed set relative to U (since the preimage of a closed set is closed for continuous functions).

If U is connected, the only simultaneously closed and open sets in U are $\emptyset$ and U , and since A by supposition is not empty, $A = U$. Hence u having interior maximizer implies u is constant in $\bar{U}$.

If U is not connected, then x belongs to a subset $V \subset U$ that is connected and for which ∂V is a non-empty subset of ∂U . Consequently $u = M$ on ∂V (so maximum is attained on the boundary of U as well as in the interior).

(c)

$$\partial_{x_i x_i} \varphi(u) = \partial_{x_i} (\varphi'(u) u_{x_i}) = \varphi''(u) (u_{x_i})^2 + \varphi'(u) u_{x_i x_i}$$

Hence

$$\Delta v = \varphi''(u) |Du|^2 \geq 0.$$

(d) $v_{x_i} = 2 \sum_{j=1}^n u_{x_j} u_{x_j x_i}$ and $v_{x_i x_i} = 2 \sum_{j=1}^n u_{x_j x_i}^2 + u_{x_j} u_{x_j x_i x_i}$. Hence

$$\Delta v = 2|D^2 u|^2 + 2 \sum_{j=1}^n \sum_{i=1}^n u_{x_j} u_{x_j x_i x_i} = 2|D^2 u|^2 + 2 \sum_{j=1}^n u_{x_j} \underbrace{\Delta u_{x_j}}_{=0} \geq 0$$

using that $\Delta u_{x_j} = \partial_{x_j} \Delta u = 0$.