

MAT4301 – Suggested exercises – week 38

September 18, 2025

1. (a) Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be sub-harmonic within a bounded, open set Ω . Prove that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

- (b) Show that the same statement with “max” replaced by “min” is not necessarily true. That is, find a sub-harmonic function u on a bounded, open set Ω for which $\min_{\overline{\Omega}} u < \min_{\partial\Omega} u$.
2. (a) (*Problem 2.5.6 in Evans*) Let Ω be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C , depending only on Ω , such that

$$\max_{\overline{\Omega}} |u| \leq C \left(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f| \right) \quad (1)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Here, $f: \overline{\Omega} \rightarrow \mathbb{R}$ and $g: \partial\Omega \rightarrow \mathbb{R}$ are given, continuous functions.

Hint: Find an $\varepsilon > 0$ so that the function $u_\varepsilon(x) = u(x) + \varepsilon|x|^2$ becomes sub-harmonic. Then do the same for the function $v_\varepsilon(x) = -u(x) + \varepsilon|x|^2$.

- (b) State and prove a result which makes the following claim rigorous: If u_1 and u_2 are solutions of (2) with data f_1, g_1 and f_2, g_2 , respectively, then u_1 and u_2 are close whenever the data are close.
3. (Relating to the proof of Theorem 2.2.14) Show that the Poisson kernel

$$K(x, y) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}_+^n} \frac{1}{|x - y|^n} dy \quad (x \in \mathbb{R}_+^n),$$

satisfies

$$\int_{\partial\mathbb{R}_+^n} K(x, y) dS(y) = 1$$

for every $x \in \mathbb{R}_+^n$. The following may be useful to simplify some technically messy steps (corrected typo in right-hand side):

$$\int_0^\infty \frac{t^{n-2}}{(1+t^2)^{n/2}} dt = \int_0^{\pi/2} \sin^{n-2} \theta d\theta = \frac{n}{2(n-1)} \frac{\alpha(n)}{\alpha(n-1)}.$$

Hints: Use that $\partial\mathbb{R}_+^n = \{y \in \mathbb{R}^n \mid y_n = 0\}$, that this is isomorphic to $\mathbb{R}^{n-1}$, and thus that the above surface integral simplifies to

$$\int_{\mathbb{R}^{n-1}} K(x, (y_1, \dots, y_{n-1}, 0)) dy_1 \cdots dy_{n-1}.$$

4. Problem 2.5.9 in Evans.