

MAT4301 – Suggested exercises – week 38

September 18, 2025

1. (a) Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be sub-harmonic within a bounded, open set Ω . Prove that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

- (b) Show that the same statement with “max” replaced by “min” is not necessarily true. That is, find a sub-harmonic function u on a bounded, open set Ω for which $\min_{\overline{\Omega}} u < \min_{\partial\Omega} u$.
2. (a) (*Problem 2.5.6 in Evans*) Let Ω be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C , depending only on Ω , such that

$$\max_{\overline{\Omega}} |u| \leq C \left(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f| \right) \quad (1)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Here, $f: \overline{\Omega} \rightarrow \mathbb{R}$ and $g: \partial\Omega \rightarrow \mathbb{R}$ are given, continuous functions.

Hint: Find an $\varepsilon > 0$ so that the function $u_\varepsilon(x) = u(x) + \varepsilon|x|^2$ becomes sub-harmonic. Then do the same for the function $v_\varepsilon(x) = -u(x) + \varepsilon|x|^2$.

- (b) State and prove a result which makes the following claim rigorous: If u_1 and u_2 are solutions of (2) with data f_1, g_1 and f_2, g_2 , respectively, then u_1 and u_2 are close whenever the data are close.
3. (Relating to the proof of Theorem 2.2.14) Show that the Poisson kernel

$$K(x, y) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}_+^n} \frac{1}{|x - y|^n} dy \quad (x \in \mathbb{R}_+^n),$$

satisfies

$$\int_{\partial\mathbb{R}_+^n} K(x, y) dS(y) = 1$$

for every $x \in \mathbb{R}_+^n$. The following may be useful to simplify some technically messy steps (corrected typo in right-hand side):

$$\int_0^\infty \frac{t^{n-2}}{(1+t^2)^{n/2}} dt = \int_0^{\pi/2} \sin^{n-2} \theta d\theta = \frac{n}{2(n-1)} \frac{\alpha(n)}{\alpha(n-1)}.$$

Hints: Use that $\partial\mathbb{R}_+^n = \{y \in \mathbb{R}^n \mid y_n = 0\}$, that this is isomorphic to $\mathbb{R}^{n-1}$, and thus that the above surface integral simplifies to

$$\int_{\mathbb{R}^{n-1}} K(x, (y_1, \dots, y_{n-1}, 0)) dy_1 \cdots dy_{n-1}.$$

4. Problem 2.5.9 in Evans.

Solution: Many solutions were kindly provided by Ulrik Skre Fjordholm.

1. (a) As in Problem 2.5.4, let $v_\varepsilon(x) = v(x) + \varepsilon|x|^2$. Then v_ε is bounded and continuous on the compact set $\overline{\Omega}$, and $\Delta v_\varepsilon(x) = \Delta v(x) + 2n\varepsilon > 0$. Moreover, by the extremal value theorem, v_ε attains a maximum at some $x_0 \in \overline{\Omega}$. If $x_0 \in \Omega$ then $\Delta v_\varepsilon(x_0) \leq 0$, contradicting $\Delta v_\varepsilon(x_0) > 0$. Hence, $x_0 \in \partial\Omega$, and the conclusion follows.
- (b) Let $\Omega = B^0(0, 1)$, the open unit ball, and let $u(x) = |x|^2$. Then $\Delta u(x) = 2n > 0$, so u is sub-harmonic, but $\min_{\overline{\Omega}} u = 0 < \min_{\partial\Omega} u = 1$.
2. (a) Let u solve (2), and define $u_\varepsilon(x) = u(x) + \varepsilon|x|^2$. Then

$$\Delta u_\varepsilon(x) = \Delta u(x) + \varepsilon \Delta |x|^2 = -f(x) + 2n\varepsilon.$$

If we choose $\varepsilon = \max_{x \in \overline{\Omega}} |f(x)|$, then $\Delta u_\varepsilon(x) \geq 0$, that is, u_ε is sub-harmonic. Hence, by Problem 1,

$$\begin{aligned} u_\varepsilon(x) &\leq \max_{\partial\Omega} u_\varepsilon = \max_{\partial\Omega} (u(x) + \varepsilon|x|^2) \leq \max_{\partial\Omega} u(x) + \varepsilon \underbrace{\max_{\partial\Omega} |x|^2}_{=R} \\ &= \max_{\partial\Omega} g + \varepsilon R \end{aligned}$$

for all $x \in \Omega$, and therefore

$$u(x) = u_\varepsilon(x) - \varepsilon|x|^2 \leq u_\varepsilon(x) \leq \max_{\partial\Omega} g + \varepsilon R.$$

Next, setting $v_\varepsilon(x) = -u(x) + \varepsilon|x|^2$, we find that v_ε is sub-harmonic for the same ε , and therefore,

$$-u(x) \leq \max_{\partial\Omega} (-g) + \varepsilon R.$$

We conclude that

$$|u(x)| = \max(u(x), -u(x)) \leq \max_{\partial\Omega} |g| + R \max_{\overline{\Omega}} |f| \leq C \left(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f| \right)$$

where $C = \max(1, R)$.

- (b) The statement is: If u_1 and u_2 are solutions of (2) with data f_1, g_1 and f_2, g_2 , respectively, then

$$\max_{\overline{\Omega}} |u_1 - u_2| \leq C \left(\max_{\partial\Omega} |g_1 - g_2| + \max_{\overline{\Omega}} |f_1 - f_2| \right).$$

For the proof, apply Problem (a) to the function $u = u_1 - u_2$, which satisfies (2) with $f = f_1 - f_2$ and $g = g_1 - g_2$. The claim is then precisely (1).

3. By the change of variables $z = (x_1, \dots, x_{n-1}) - (y_1, \dots, y_{n-1})$ on $\mathbb{R}^{n-1}$ and shifting to polar coordinates, we obtain

$$\begin{aligned} I &= \int_{\mathbb{R}^{n-1}} K(x, (y_1, \dots, y_{n-1}, 0)) dy_1 \cdots dy_{n-1} \\ &= \frac{2x_n}{n\alpha(n)} \int_{\mathbb{R}^{n-1}} \frac{1}{(|z|^2 + x_n^2)^{n/2}} dz \\ &= \frac{2x_n}{n\alpha(n)} \int_0^\infty (n-1)\alpha(n-1) \frac{r^{n-2}}{(r^2 + x_n^2)^{n/2}} dr. \end{aligned}$$

Change of variables $tx_n = r$ further yields

$$I = \frac{2(n-1)\alpha(n-1)}{n\alpha(n)} \int_0^\infty \frac{t^{n-2}}{(1+t^2)^{n/2}} dt = 1,$$

where the last line follows from the formula given in the problem formulation.

4. Following the hint and recalling the Green's function formula for the solution on the half space, we consider

$$\begin{aligned} \frac{u(\lambda e_n) - u(0)}{\lambda} &= \frac{u(\lambda e_n) - 0}{\lambda} \\ &= \int_{\partial \mathbb{R}_+^n} |y| \frac{K(\lambda e_n, y)}{\lambda} dS(y) \\ &= \frac{2\lambda}{n\alpha(n)} \int_{\partial \mathbb{R}^{n-1}} \frac{|(y_1, \dots, y_{n-1}, 0)|}{\lambda |(y_1, \dots, y_{n-1}, 0) - (0, \dots, 0, \lambda)|^n} dy_1 \cdots dy_{n-1} \\ &= \frac{2}{n\alpha(n)} \int_{\partial \mathbb{R}^{n-1}} \frac{|z|}{(|z|^2 + \lambda^2)^{n/2}} dz \\ &= \frac{2(n-1)\alpha(n-1)}{n\alpha(n)} \int_0^\infty \frac{r}{(r^2 + \lambda^2)^{n/2}} dz \\ &= \frac{2(n-1)\alpha(n-1)}{n\alpha(n)} \times \begin{cases} [\log(r^2 + \lambda^2)/2]_{r=0}^\infty & n = 2 \\ \left[(r^2 + \lambda^2)^{1-n/2} / (2(-n/2 + 1)) \right]_{r=0}^\infty & n \geq 3 \end{cases} \end{aligned}$$

For all $n \geq 2$, we conclude that

$$\frac{u(\lambda e_n) - u(0)}{\lambda} \rightarrow \infty \quad \text{as } \lambda \downarrow 0.$$