

MAT4301 – Suggested exercises – week 39

September 27, 2025

1. (*Maximum principle*) Let $\Omega \subset \mathbb{R}^n$ be open and bounded and consider the *advection-diffusion problem*

$$b \cdot Du = \mu \Delta u \quad (\text{in } \Omega) \quad (1)$$

where $b \in \mathbb{R}^n$ is a fixed vector (the *velocity*) and $\mu > 0$ is a given number (the *diffusion coefficient* or *viscosity*). Prove the maximum principle

$$u(x) \leq \max_{y \in \partial\Omega} u(y) \quad \forall x \in \Omega \quad (2)$$

for any function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying (1).

Hint: Show first that if a function v satisfies $b \cdot Dv < \mu \Delta v$ in Ω , then v cannot have a maximum in Ω . Then let $v_\varepsilon(x) = u(x) + \varepsilon(|x|^2 - Mb \cdot x)$ for some $M, \varepsilon > 0$.

Remark. Consider the time-dependent equation

$$v_t + b \cdot Dv = \mu \Delta v \quad (1')$$

for some $v = v(x, t)$. If v is a stationary solution (i.e., time-independent) then $v_t \equiv 0$ and so $u(x) := v(x, 0)$ will be a solution of (1). The PDE (1') features both transport ($b \cdot Dv$) and diffusion ($\mu \Delta v$). For instance, it can model the distribution of heat $v(x, t)$ in a fluid which has heat conductivity μ and which moves through space with velocity b .

2. (*Problem 2.5.12 from Evans*) Suppose u is smooth and solves $u_t = \Delta u$ in $\mathbb{R}^n \times (0, \infty)$.

(a) Show that $u^\lambda(x, t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) Use (a) to show that $v(x, t) := x \cdot Du(x, t) + 2tu_t(x, t)$ solves the heat equation as well.

Hint: Compute $\frac{\partial}{\partial \lambda} u^\lambda(x, t)$.

3. (*Problem 2.5.14 from Evans*) Write down an explicit formula for a solution of

$$\begin{cases} u_t + cu = \Delta u + f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \quad (3)$$

where $c \in \mathbb{R}$ is a constant.

Hint: Pretend first that the Δu term is not there and multiply the equation by an integrating factor. Reduce the problem to one for which you already know the solution.

Remark. The term cu is a *reaction term* – it will “produce” or “remove” heat at a rate proportional to $u(x, t)$. For instance, u could be the temperature distribution in a reactive chemical: The reaction produces heat, and the rate of reaction is proportional to the temperature. The term f is a *source term* – it similarly “produces” or “removes” heat, but irrespective of the value of u .

4. Let $r > 0$ and consider the the heat ball

$$E(x, t; r) = \{(y, s) \in \mathbb{R}^{n+1} \mid s < t, \Phi(x - y, t - s) \geq r^{-n}\},$$

where $\Phi(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$ denotes the fundamental solution of the heat equation. In connection with the proof of Theorem 2.3.3 in Evans, show that

$$I = \frac{1}{4r^n} \iint_{E(r)} \frac{|y|^2}{s^2} dy ds = 1,$$

where $E(r) := E(0, 0; r)$.

Hint: The integrand is non-negative, so Tonelli’s theorem applies. Note that $(y, s) \in E(r) \iff 0 > s \geq -r^2 e^{-|y|^2/2ns}/4\pi$, so that

$$I = \frac{1}{4r^n} \int_{\mathbb{R}^n} \int_{-\infty}^0 \mathbb{1}_{s \geq -r^2 e^{-|y|^2/2ns}/4\pi} \frac{|y|^2}{s^2} ds dy.$$

Try to find a change of variables of y such that the right-hand side of $s \geq -r^2 e^{-|y|^2/2ns}/4\pi$ does not depend on s .

Lastly, the following result may also come handy

$$(4\pi)^{-n/2} \int_{\mathbb{R}^n} |y|^2 e^{-|y|^2/4} = 2n.$$

(This follows from integration by parts, or, if you are probabilistically inclined, the fact, unrelated to this course, that $\rho(y) = (4\pi)^{-n/2} e^{-|y|^2/4}$ is the probability density function of a multivariate normal distribution $X \sim N(0, \sqrt{2}I_n)$ and that the above equals the expected value of $|X|^2$.)

Solution: Many solutions were kindly provided by Ulrik Skre Fjordholm.

1. Assume first that $v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $b \cdot Dv < \mu \Delta v$ in Ω . If $x_0 \in \Omega$ is a maximum of v , then $Dv(x_0) = 0$ and $\Delta v(x_0) \leq 0$. Thus,

$$0 > b \cdot Dv(x_0) - \mu \Delta v(x_0) \geq 0,$$

a contradiction.

Next, as indicated in the hint, let $v_\varepsilon(x) = u(x) + \varepsilon(|x|^2 - Mb \cdot x)$ for some $\varepsilon, M > 0$. We know that $b \cdot Du - \mu \Delta u = 0$, so

$$\begin{aligned} b \cdot Dv_\varepsilon - \mu \Delta v_\varepsilon &= b \cdot Du - \mu \Delta u + \varepsilon b \cdot (2x - Mb) - 2\varepsilon \mu \\ &= \varepsilon(2b \cdot x - M|b|^2 - 2\mu). \end{aligned}$$

If we let $\varepsilon > 0$ be arbitrary, and pick M so large that $2b \cdot x \leq M|b|^2$ for every $x \in \Omega$ (this is possible since Ω is bounded), then the above is strictly

negative. Hence, v_ε cannot have a maximum in Ω ; in particular,

$$v_\varepsilon(x) \leq \max_{\partial\Omega} v_\varepsilon.$$

Letting $\varepsilon \rightarrow 0$ and noting that $v_\varepsilon \rightarrow u$ uniformly, we conclude the proof of (2).

2. (a) We have

$$u_t^\lambda(x, t) - \Delta u^\lambda(x, t) = \lambda^2 u_t(\lambda x, \lambda^2 t) - \lambda^2 \Delta u(\lambda x, \lambda^2 t) = 0.$$

(b) We follow the hint and get

$$\frac{\partial}{\partial \lambda} u^\lambda(x, t) = x \cdot Du(\lambda x, \lambda^2 t) + 2\lambda u_t(\lambda x, \lambda^2 t).$$

Comparing with the definition of v , we see that $v = \frac{\partial}{\partial \lambda} u^\lambda(x, t)|_{\lambda=1}$. Since u^λ solves the heat equation for any $\lambda > 0$, also its derivative with respect to λ will solve the heat equation (since we can interchange the order of differentiation). Thus, also v solves the heat equation.

3. We follow the hint and multiply the PDE by the integrating factor e^{-ct} to get

$$\frac{\partial}{\partial t} (u(x, t) e^{-ct}) = e^{-ct} \Delta u(x, t) = e^{-ct} f(x, t).$$

Define $v(x, t) = u(x, t) e^{-ct}$. Then $e^{-ct} \Delta u(x, t) = \Delta v(x, t)$. Hence, if $h(x, t) := e^{-ct} f(x, t)$, we see that v solves the inhomogeneous heat equation

$$v_t = \Delta v + h$$

with initial data $v(x, 0) = e^0 u(x, 0) = g(x)$. Duhamel's principle yields the solution

$$v(x, t) = \frac{1}{\sqrt{4\pi t}^n} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y) dy + \int_0^t \int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/4(t-s)}}{\sqrt{4\pi(t-s)}^n} h(y, s) dy ds.$$

Finally, inserting $u(x, t) = e^{ct} v(x, t)$ and $h(x, s) = e^{-cs} f(x, s)$, we get the formula

$$u(x, t) = \frac{e^{ct}}{\sqrt{4\pi t}^n} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y) dy + \int_0^t \int_{\mathbb{R}^n} e^{c(t-s)} \frac{e^{-|x-y|^2/4(t-s)}}{\sqrt{4\pi(t-s)}^n} f(y, s) dy ds.$$

4. We follow the hint with the change of variables $\tau = -s$ to obtain

$$I = \frac{1}{4r^n} \int_{\mathbb{R}^n} \int_0^\infty \mathbb{1}_{\tau \leq r^2 e^{|y|^2/2n\tau}/4\pi} \frac{|y|^2}{\tau^2} d\tau dy.$$

Next, the change of variables $z = y/\sqrt{\tau}$ yields

$$\begin{aligned}
 I &= \frac{1}{4r^n} \int_{\mathbb{R}^n} \int_0^\infty \mathbb{1}_{\tau \leq r^2 e^{|z|^2/2n}/4\pi} |z|^2 \tau^{n/2-1} d\tau dz \\
 &= \frac{1}{4r^n} \int_{\mathbb{R}^n} |z|^2 \int_0^{r^2 e^{|z|^2/2n}/4\pi} \tau^{n/2-1} d\tau dz \\
 &= \frac{1}{2n(4\pi)^{n/2}} \int_{\mathbb{R}^n} |z|^2 e^{|z|^2/4} dz \\
 &= 1 .
 \end{aligned}$$