

MAT4301 – Suggested exercises – week 41

October 2, 2025

1. (a) Write down d'Alembert's formula for the solution of the one-dimensional wave equation with wave speed $c > 0$,

$$\begin{cases} u_{tt} = c^2 u_{xx} & (x \in \mathbb{R}, t > 0) \\ u(x, 0) = g(x) \\ u_t(x, 0) = h(x). \end{cases} \quad (1)$$

Draw the backwards wave cone from a point $(x, t) \in \mathbb{R} \times \mathbb{R}_+$.

Remark. The backwards wave cone at (x, t) is the set of points (y, s) with $s < t$ such that the solution at (x, t) depends on the solution at (y, s) .

- (b) Solve (1) when $c = 1$, $g(x) = \sin(x)$, $h(x) = \cos(2x)$.
2. Consider the one-dimensional wave equation on $\mathbb{R}_+$ with *Neumann boundary condition*,

$$\begin{cases} u_{tt} = c^2 u_{xx} & (x > 0, t > 0) \\ u(x, 0) = g(x) & (x > 0) \\ u_t(x, 0) = h(x) & (x > 0) \\ u_x(0, t) = 0 & (t > 0). \end{cases} \quad (2)$$

Where $g \in C^2([0, \infty))$ and $h \in C^1([0, \infty))$ are given functions satisfying $g'(0) = h'(0) = 0$.

- (a) Find a solution formula for (2).
Hint: See [p. 69, Evans]. Make an even reflection.
- (b) Make a physical interpretation of the boundary condition and of your solution formula.
3. (a) Find a solution formula for the initial value problem

$$\begin{cases} u_t = k \Delta u & (x \in \mathbb{R}^n, t > 0) \\ u(x, 0) = g(x) & (x \in \mathbb{R}^n) \end{cases} \quad (3)$$

for some $k > 0$ and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Note: k is the heat conductivity, indicating how quickly heat is diffused through the material.

- (b) What would happen for negative k ?

(c) Find a solution formula for the initial value problem

$$\begin{cases} u_t(x, t) = k(t)\Delta u(x, t) & (x \in \mathbb{R}^n, t > 0) \\ u(x, 0) = g(x) & (x \in \mathbb{R}^n) \end{cases} \quad (4)$$

for a continuous function $k: [0, \infty) \rightarrow (0, \infty)$.

4. (Part of the proof of Theorem 2.3.6) Let $T, \varepsilon > 0$. Show that the function

$$w(x, t) = \frac{1}{(T + \varepsilon - t)^{n/2}} e^{|x|^2/4(T+\varepsilon-t)}$$

satisfies the heat equation $w_t = \Delta w$ for $t \in (0, T]$, $x \in \mathbb{R}^n$. Note that $w(x, t)$ increases very quickly as $|x| \rightarrow \infty$.

5. Consider the Cauchy problem for the heat equation

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R} \times (0, \infty) \\ u(x, 0) = 0 & \text{for } x \in \mathbb{R}. \end{cases} \quad (5)$$

Clearly, the trivial solution $u \equiv 0$ is one solution of (5), and this is also the solution we would get from the solution formula $u(t) = \Phi(\cdot, t) * u(\cdot, 0)$.

(a) Let now $\alpha > 1$ and define

$$v(x, t) := \sum_{k=0}^{\infty} \frac{2^k g^{(k)}(t)}{(2k)!} x^{2k}, \quad g(t) := \begin{cases} e^{-1/t^2} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases} \quad (6)$$

where $g^{(k)}$ is the k -th derivative of g . Show either by formal computations or rigorously that v also solves (5).

Hint: For a rigorous argument, you may use that there exists a constant $C > 0$ s.t.

$$\left| \frac{d^j}{dt^j} g(t) \right| = |g^{(j)}(t)| \leq \frac{(j!)^{3/2}}{(Ct)^j} e^{-1/t^2} \quad \text{for all } t > 0$$

to verify that derivatives and sums can be interchanged and for studying the limit of $u(x, t)$ when $t \rightarrow 0$. (The inequality can be shown for such Gevrey-class functions as g through a complex-analysis argument.)

(b) Explain why there are infinitely many solutions to the Cauchy problem for the heat equation. Speculate on how the above result fits with our “conditional uniqueness” result (Theorem 7 in Section 2.3 of Evans).

For those interested (not curriculum), here is some MATLAB code for a very rough study of the magnitude of $v(x, t)$ versus $\exp(a|x|)$ through numerical comparison of the magnitude of the power-series terms of $v(x, t)$ to the even power series terms of $\exp(a|x|)$ (copy, paste, run and play around with other parameter values to study growth of v):

```

clear; clc;
syms s
f_sym = exp(-1/s^2);

t = 1;      % fixed t
x = 20;     % fixed x
N = 400;    % number of terms
a = 10;     % exponential rate for comparison
%symbolic representation of scaled version of g(t) for
%exact differentiation
G = f_sym;

T = zeros(1,N+1); %Terms in power series of v(x,t)
c = zeros(1,N+1);
T(1) = 1;
c(1) = double(subs(G,s,t));

for j = 1:N
    if j <= N
        G = 2*diff(G,s)/sqrt((2*j-1)*(2*j));
    end
    c(j+1) = double(subs(G,s,t));

    T(j+1) = T(j) .* x^2 ./ sqrt((2*j-1)*(2*j));
end
T = T.*c;

C_exp = zeros(1,N+1); % even terms in power series of exp(a|x|)
C_exp(1) = 1; % j=0
for j = 1:N
    C_exp(j+1) = C_exp(j) * (a*x)^2 / ((2*j-1)*(2*j));
end

figure;
stem(0:N, abs(T), 'b', 'filled'); hold on;
stem(0:N, C_exp, 'r--', 'filled');
set(gca,'YScale','log');
xlabel('j');
ylabel('Magnitude of power-series terms (log scale)');
legend('Terms of v(x,t)', 'even terms of exp(a|x|)');
title(['Comparison of power-series terms at t=', ...
    num2str(t), ', x=', num2str(x)]);
grid on;

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