

MAT4301 – Suggested exercises – week 41

October 5, 2025

1. (a) Write down d'Alembert's formula for the solution of the one-dimensional wave equation with wave speed $c > 0$,

$$\begin{cases} u_{tt} = c^2 u_{xx} & (x \in \mathbb{R}, t > 0) \\ u(x, 0) = g(x) \\ u_t(x, 0) = h(x). \end{cases} \quad (1)$$

Draw the backwards wave cone from a point $(x, t) \in \mathbb{R} \times \mathbb{R}_+$.

Remark. The backwards wave cone at (x, t) is the set of points (y, s) with $s < t$ such that the solution at (x, t) depends on the solution at (y, s) .

- (b) Solve (1) when $c = 1$, $g(x) = \sin(x)$, $h(x) = \cos(2x)$.
2. Consider the one-dimensional wave equation on $\mathbb{R}_+$ with *Neumann boundary condition*,

$$\begin{cases} u_{tt} = c^2 u_{xx} & (x > 0, t > 0) \\ u(x, 0) = g(x) & (x > 0) \\ u_t(x, 0) = h(x) & (x > 0) \\ u_x(0, t) = 0 & (t > 0). \end{cases} \quad (2)$$

Where $g \in C^2([0, \infty))$ and $h \in C^1([0, \infty))$ are given functions satisfying $g'(0) = h'(0) = 0$.

- (a) Find a solution formula for (2).
Hint: See [p. 69, Evans]. Make an even reflection.
- (b) Make a physical interpretation of the boundary condition and of your solution formula.
3. (a) Find a solution formula for the initial value problem

$$\begin{cases} u_t = k \Delta u & (x \in \mathbb{R}^n, t > 0) \\ u(x, 0) = g(x) & (x \in \mathbb{R}^n) \end{cases} \quad (3)$$

for some $k > 0$ and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Note: k is the heat conductivity, indicating how quickly heat is diffused through the material.

- (b) What would happen for negative k ?

(c) Find a solution formula for the initial value problem

$$\begin{cases} u_t(x, t) = k(t)\Delta u(x, t) & (x \in \mathbb{R}^n, t > 0) \\ u(x, 0) = g(x) & (x \in \mathbb{R}^n) \end{cases} \quad (4)$$

for a continuous function $k: [0, \infty) \rightarrow (0, \infty)$.

4. (Part of the proof of Theorem 2.3.6) Let $T, \varepsilon > 0$. Show that the function

$$w(x, t) = \frac{1}{(T + \varepsilon - t)^{n/2}} e^{|x|^2/4(T+\varepsilon-t)}$$

satisfies the heat equation $w_t = \Delta w$ for $t \in (0, T]$, $x \in \mathbb{R}^n$. Note that $w(x, t)$ increases very quickly as $|x| \rightarrow \infty$.

5. Consider the Cauchy problem for the heat equation

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R} \times (0, \infty) \\ u(x, 0) = 0 & \text{for } x \in \mathbb{R}. \end{cases} \quad (5)$$

Clearly, the trivial solution $u \equiv 0$ is one solution of (5), and this is also the solution we would get from the solution formula $u(t) = \Phi(\cdot, t) * u(\cdot, 0)$.

(a) Let

$$v(x, t) := \sum_{k=0}^{\infty} \frac{2^k g^{(k)}(t)}{(2k)!} x^{2k}, \quad g(t) := \begin{cases} e^{-1/t^2} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases} \quad (6)$$

where $g^{(k)}$ is the k -th derivative of g . Show either by formal computations or rigorously that v also solves (5).

Hint: For a rigorous argument, you may use that there exists a constant $C > 0$ s.t.

$$\left| \frac{d^j}{dt^j} g(t) \right| = |g^{(j)}(t)| \leq \frac{(j!)^{3/2}}{(Ct)^j} e^{-1/t^2} \quad \text{for all } t > 0$$

to verify that derivatives and sums can be interchanged and for studying the limit of $u(x, t)$ when $t \rightarrow 0$. (The inequality can be shown for such Gevrey-class functions as g through a complex-analysis argument.)

(b) Explain why there are infinitely many solutions to the Cauchy problem for the heat equation. Speculate on how the above result fits with our “conditional uniqueness” result (Theorem 7 in Section 2.3 of Evans).

For those interested (not curriculum), here is some MATLAB code for a very rough study of the magnitude of $v(x, t)$ versus $\exp(a|x|)$ through numerical comparison of the magnitude of the power-series terms of $v(x, t)$ to the even power series terms of $\exp(a|x|)$ (copy, paste, run and play around with other parameter values to study growth of v):

```

clear; clc;
syms s
f_sym = exp(-1/s^2);

t = 1;      % fixed t
x = 20;     % fixed x
N = 400;    % number of terms
a = 10;     % exponential rate for comparison
%symbolic representation of scaled version of g(t) for
%exact differentiation
G = f_sym;

T = zeros(1,N+1); %Terms in power series of v(x,t)
c = zeros(1,N+1);
T(1) = 1;
c(1) = double(subs(G,s,t));

for j = 1:N
    if j <= N
        G = 2*diff(G,s)/sqrt((2*j-1)*(2*j));
    end
    c(j+1) = double(subs(G,s,t));

    T(j+1) = T(j) .* x^2 ./ sqrt((2*j-1)*(2*j));
end
T = T.*c;

C_exp = zeros(1,N+1); % even terms in power series of exp(a|x|)
C_exp(1) = 1; % j=0
for j = 1:N
    C_exp(j+1) = C_exp(j) * (a*x)^2 / ((2*j-1)*(2*j));
end

figure;
stem(0:N, abs(T), 'b', 'filled'); hold on;
stem(0:N, C_exp, 'r--', 'filled');
set(gca,'YScale','log');
xlabel('j');
ylabel('Magnitude of power-series terms (log scale)');
legend('Terms of v(x,t)', 'even terms of exp(a|x|)');
title(['Comparison of power-series terms at t=', ...
    num2str(t), ', x=', num2str(x)]);
grid on;

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Solution: Many solutions were kindly provided by Ulrik Skre Fjordholm.

1. (a) If $v(x, t) := u(x, t/c)$ then $v_{tt}(x, t) = \frac{1}{c^2}u_{tt}(x, t/c)$, $v_{xx}(x, t) = u_{xx}(x, t/c)$, so v solves

$$\begin{cases} v_{tt} = v_{xx} & (x \in \mathbb{R}, t > 0) \\ v(x, 0) = g(x) \\ v_t(x, 0) = \frac{1}{c}h(x). \end{cases}$$

Hence, according to d'Alembert's formula for the case $c = 1$, we get

$$v(x, t) = \frac{1}{2} \int_{x-t}^{x+t} \frac{1}{c} h(y) dy + \frac{1}{2} (g(x-t) + g(x+t))$$

and therefore

$$u(x, t) = v(x, ct) = \frac{1}{2c} \int_{x-ct}^{x+ct} h(y) dy + \frac{1}{2} (g(x-ct) + g(x+ct)).$$

(b) We get

$$\begin{aligned} u(x, t) &= \frac{1}{2} \int_{x-t}^{x+t} \cos(2y) dy + \frac{1}{2} (\sin(x-t) + \sin(x+t)) \\ &= \frac{1}{4} (\sin(2(x+t)) - \sin(2(x-t))) + \frac{1}{2} (\sin(x-t) + \sin(x+t)). \end{aligned}$$

This corresponds to two sets of waves with frequencies $\frac{1}{\pi}$ and $\frac{1}{2\pi}$, respectively, moving in both directions at speed ± 1 .

2. (a) Assume that u solves (2) and define $v: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ by the even reflection

$$v(x, t) = u(|x|, t) = \begin{cases} u(x, t) & x \geq 0 \\ u(-x, t) & x < 0. \end{cases}$$

Then $v_{tt}(x, t) = u_{tt}(|x|, t)$, $v_x(x, t) = \text{sgn}(x)u_x(x, t)$, $v_{xx}(x, t) = \delta(x)u_x(|x|, t) + \text{sgn}(x)^2 u_{xx}(|x|, t) = u_{xx}(|x|, t)$. Here, $\text{sgn}(x)$ is the sign function, defined as 1 when $x \geq 0$ and -1 when $x < 0$. Its (weak) derivative is δ , the Dirac measure, which is zero away from the origin and “ ∞ ” at the origin; since $u_x(0, t) = 0$, the term δu_x is zero. Hence, v satisfies

$$\begin{cases} v_{tt} = c^2 v_{xx} & (x \in \mathbb{R}, t > 0) \\ v(x, 0) = \bar{g}(x) & (x \in \mathbb{R}) \\ v_t(x, 0) = \bar{h}(x) & (x \in \mathbb{R}) \\ v_x(0, t) = 0 & (t > 0), \end{cases}$$

where $\bar{g}$ and $\bar{h}$ are the even extensions of g and h , respectively. Hence,

$$\begin{aligned} v(x, t) &= \frac{1}{2c} \int_{x-ct}^{x+ct} \bar{h}(y) dy + \frac{1}{2} (\bar{g}(x-ct) + \bar{g}(x+ct)) \\ &= \frac{1}{2c} \int_{x-ct}^{x+ct} h(|y|) dy + \frac{1}{2} (g(|x-ct|) + g(|x+ct|)). \end{aligned}$$

(Note that $v_x(0, t) = 0$, so v indeed satisfies the fourth condition above.) We finally obtain for all $x, t \geq 0$

$$u(x, t) = v(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} h(|y|) dy + \frac{1}{2} (g(|x-ct|) + g(x+ct)).$$

(Since $x, t, c \geq 0$ we have $|x+ct| = x+ct$.)

- (b) If $u(x, t)$ is the vertical displacement of a vibrating string at position x at time t , then $u_x(x, t) = 0$ means that the string is lying horizontally at (x, t) . Thus, the Neumann boundary condition in (2) asserts that at although the left endpoint of the string might be moving over time, it should lie flat at all times.

Note that if $x-ct > 0$ then the above solution formula reduces to the

standard d'Alembert formula, while if $x - ct < 0$ then we get

$$\begin{aligned} u(x, t) &= \frac{1}{2c} \left(\int_{x-ct}^0 h(|y|) dy + \int_0^{x+ct} h(|y|) dy \right) + \frac{1}{2} (g(|x-ct|) + g(x+ct)) \\ &= \frac{1}{2c} \left(\int_{x-ct}^0 h(-y) dy + \int_0^{x+ct} h(y) dy \right) + \frac{1}{2} (g(ct-x) + g(x+ct)) \\ &= \frac{1}{2c} \left(2 \int_0^{ct-x} h(y) dy + \int_{ct-x}^{x+ct} h(y) dy \right) + \frac{1}{2} (g(ct-x) + g(x+ct)). \end{aligned}$$

Thus, the velocity data h is counted twice in the interval $[0, ct-x]$, while the displacement data g is reflected from the point $x-ct < 0$ to the point $ct-x > 0$. We can interpret this as saying that waves are reflected from the “wall” $x = 0$ and bounce back into the domain.

3. (a) If $k < 0$, our formula would involve the irrational term $\sqrt{4\pi kt}$, and we would be integrating e to the power of $|x-y|^2/4|k|t$, which is infinite. Thus, the formula clearly breaks down.

To interpret this, note that the change of variables $(x, t) \mapsto (x, t/k)$ — which transforms the equation to the standard heat equation — involves flipping the direction of time (t decreases instead of increases). This is an indication that solving the heat equation *backwards in time* is a bad idea.

- (b) Let $\tau: [0, \infty) \rightarrow [0, \infty)$ be some function, to be determined, and let $v(x, t) = u(x, \tau(t))$. Then

$$v_t(x, t) = \tau'(t)u_t(x, \tau(t)) = \tau'(t)k(t)\Delta u(x, \tau(t)) = \tau'(t)k(t)\Delta v(x, t).$$

Thus, if $\tau'(t) = 1/k(t)$, that is, $\tau(t) = C + \int_0^t 1/k(s) ds$, then v solves the standard heat equation. Its initial data is $v(x, 0) = u(x, \tau(0)) = u(x, C)$, so if we choose $C = 0$ then $\tau(0) = 0$, and $v(x, 0) = u(x, 0) = g(x)$. Inverting the definition of v , we get $u(x, t) = v(x, \tau^{-1}(t))$, where τ^{-1} is the inverse of τ (which we cannot compute explicitly without knowing what k is). To conclude, we get the solution formula

$$u(x, t) = \frac{1}{\sqrt{4\pi\tau^{-1}(t)}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4\tau^{-1}(t)} g(y) dy.$$

4. This is solved by direct calculation.

5. (a) Differentiating, we formally have

$$v_t(x, t) = \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k}$$

and

$$\begin{aligned} v_x(x, t) &= \sum_{k=1}^{\infty} \frac{2^k g^{(k)}(t)}{(2k)!} 2k x^{2k-1}, \\ v_{xx}(x, t) &= \sum_{k=1}^{\infty} \frac{2^k g^{(k)}(t)}{(2k)!} 2k(2k-1) x^{2k-2} = \sum_{k=1}^{\infty} \frac{2^k g^{(k)}(t)}{(2(k-1))!} x^{2(k-1)} \\ &= \sum_{k=0}^{\infty} \frac{2^k g^{(k+1)}(t)}{(2k)!} x^{2k}, \end{aligned}$$

so $v_t = v_{xx}$. And using that $\lim_{t \rightarrow 0} g^{(k)}(t) = 0$, we formally have that $\lim_{t \rightarrow 0} v(x, t) = \sum_{k=0}^{\infty} \lim_{t \rightarrow 0} \frac{2^k g^{(k)}(t)}{(2k)!} x^{2k} = 0$.

Sketch of rigorous proof: Since $(2k)! \geq 2^k (k!)^2$, we have that

$$a_k = \frac{2^k (k!)^{3/2}}{(2k)!} \leq \frac{1}{\sqrt{k!}},$$

and, by the hint,

$$c_k(x, t) = \frac{2^k g^{(k)}(t)}{(2k)!} x^{2k} e^{-1/t^2} \leq \frac{1}{\sqrt{k!}} \left(\frac{x^2}{Ct} \right)^k e^{-1/t^2}.$$

Hölder's inequality yields that

$$|v(x, t)| \leq e^{-1/(2t^2)} \sqrt{\sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} \left(\frac{x^4}{C^2} \right)^k} \sqrt{e^{-1/t^2} \sum_{k=0}^{\infty} \frac{1}{k!} t^{-2k}} = e^{x^4/C^2 - 1/(2t^2)} \rightarrow 0$$

as $t \rightarrow 0$.

To justify interchange of derivation limits and sum, as needed in the formal calculation above when proving that $v_t = v_{xx}$, it suffices to show that on any compact set $U_R = [-R, R] \times [a, b]$ with $R > 0$ and $b > a > 0$, the respective sums of the sequences

$$(c_k(x, t)), (\partial_t c_k(x, t)), (\partial_x c_k(x, t)), (\partial_{xx} c_k(x, t)) \subset C^\infty(U_{R,T})$$

all converge uniformly (on U_R) and apply Weierstrass M-test and the termwise differentiation theorem, as illustrated for v_t below.

First note that $|c_k(x, t)| \leq \frac{1}{\sqrt{k!}} \left(\frac{R^2}{Ca} \right)^k e^{-1/a^2} =: m_k$ on U_R , which, as m_k is absolutely summable (apply Hölder), implies that $S_n(x, t) = \sum_{k=0}^n c_k(x, t)$ converges uniformly on U_R , as

$$|v(x, t) - S_n(x, t)| \leq \sum_{k=n+1}^{\infty} m_k \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let (m_k) now be a sequence that is absolutely summable and satisfies

$$\max(|c_k(x, t)|, |\partial_t c_k(x, t)|, |\partial_x c_k(x, t)|, |\partial_{xx} c_k(x, t)|) \leq m_k$$

for all $(x, t) \in U_T$. (such a sequence can be found similarly as for (c_k) above). Then for any interior point (x, t) of U_R any $h \in \mathbb{R}$ sufficiently small the mean-value theorem implies that there exists a sequence $(\theta_k) \subset [0, 1]$ s.t. for any $n \in \mathbb{N}$,

$$\begin{aligned} |v_t(x, t) - \sum_{k=0}^{\infty} \partial_t c_k(x, t)| &= \lim_{h \rightarrow 0} \left| \frac{v(x, t+h) - v(x, t)}{h} - \sum_{k=0}^{\infty} \partial_t c_k(x, t) \right| \\ &= \lim_{h \rightarrow 0} \left| \sum_{k=0}^{\infty} \frac{c_k(x, t+h) - c_k(x, t)}{h} - \partial_t c_k(x, t) \right| \\ &\leq \lim_{h \rightarrow 0} \sum_{k=0}^{\infty} |\partial_t c_k(x, t + \theta_k h) - \partial_t c_k(x, t)| \\ &\leq \lim_{h \rightarrow 0} \sum_{k=0}^n |\partial_t c_k(x, t + \theta_k h) - \partial_t c_k(x, t)| + 2 \sum_{k=n+1}^{\infty} m_k \\ &= \sum_{k=0}^n \lim_{h \rightarrow \infty} |\partial_t c_k(x, t + \theta_k h) - \partial_t c_k(x, t)| + 2 \sum_{k=n+1}^{\infty} m_k \\ &= 2 \sum_{k=n+1}^{\infty} m_k \end{aligned}$$

Taking the limit $n \rightarrow \infty$ yields that $v_t(x, t) = \sum_{k=0}^{\infty} \partial_t c_k(x, t)$.

Similar reasoning shows that the other interchanges of derivatives and summation also hold.

- (b) If u is a solution to the heat equation with initial data $u(x, 0) = f(x)$, then the function $u(x, t) + \alpha v(x, t)$ is a solution to the same problem, for any $\alpha \in \mathbb{R}$. Thus, there are infinitely many solutions.

It turns out that v does not satisfy the uniform upper bound $|v(x, t)| \leq Ae^{a|x|^2}$ imposed in the uniqueness result, so our observations do not contradict the uniqueness result.