

MAT4301 – Suggested exercises – week 41

October 10, 2025

1. (Small simplification of Problem 2.5.17 in Evans) We say that $v \in C_1^2(U_T)$ is a subsolution of the heat equation if $v_t - \Delta v \leq 0$ in U_T .

- (a) Suppose that v is a smooth subsolution. Following the proof of Theorem 2.3.3, it holds for sufficiently small $r > 0$,
 $\psi(y, s; r) := -\frac{n}{2} \log(-4\pi s) + |y|^2/4s + n \log r$ and

$$\varphi(r) = \frac{1}{r^n} \iint_{E(r)} v(y, s) \frac{|y|^2}{s^2} dy ds$$

that

$$\varphi'(r) = \sum_{i=1}^n \frac{1}{r^{n+1}} \iint_{E(r)} -4nv_s \psi - \frac{2n}{s} v_{y_i} y_i dy ds$$

Use this and the fact that

$$\frac{1}{4r^n} \iint_{E(r)} \frac{|y|^2}{s^2} dy ds = 1 \quad \forall r > 0,$$

to show that

$$v(x, t) \leq \frac{1}{4r^n} \iint_{E(x, t; r)} v(y, s) \frac{|x - y|^2}{(t - s)^2} dy ds$$

for all $E(x, t; r) \subset U$.

- (b) Prove therefore that $\max_{\overline{U}_T} v = \max_{\Gamma_T} v$.
 (c) Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be smooth, convex. Assume that u is smooth and solves the heat equation in U_T . Prove that $v := F(u)$ then is a subsolution.
 (d) Prove that $v = |Du|^2 + u_t^2$ is a subsolution whenever u is smooth and solves the heat equation in U_T .
2. Problem 2.5.21 in Evans. As always when smoothness is not explicitly stated, assume all functions considered are smooth.

Hint: For any vector field $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, recall that $\text{curl} A(x) := \nabla \times A(x)$ and the curl-curl identity tells us that under sufficient regularity,

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \Delta A,$$

(where ∇ here plays the same role as the gradient operator D does normally in Evans' book).

3. Problem 3.5.1 in Evans.