

MAT4301 – Suggested exercises – week 41

October 10, 2025

1. (Small simplification of Problem 2.5.17 in Evans) We say that $v \in C_1^2(U_T)$ is a subsolution of the heat equation if $v_t - \Delta v \leq 0$ in U_T .

- (a) Suppose that v is a smooth subsolution. Following the proof of Theorem 2.3.3, it holds for sufficiently small $r > 0$,
 $\psi(y, s; r) := -\frac{n}{2} \log(-4\pi s) + |y|^2/4s + n \log r$ and

$$\varphi(r) = \frac{1}{r^n} \iint_{E(r)} v(y, s) \frac{|y|^2}{s^2} dy ds$$

that

$$\varphi'(r) = \sum_{i=1}^n \frac{1}{r^{n+1}} \iint_{E(r)} -4nv_s \psi - \frac{2n}{s} v_{y_i} y_i dy ds$$

Use this and the fact that

$$\frac{1}{4r^n} \iint_{E(r)} \frac{|y|^2}{s^2} dy ds = 1 \quad \forall r > 0,$$

to show that

$$v(x, t) \leq \frac{1}{4r^n} \iint_{E(x, t; r)} v(y, s) \frac{|x - y|^2}{(t - s)^2} dy ds$$

for all $E(x, t; r) \subset U$.

- (b) Prove therefore that $\max_{\overline{U_T}} v = \max_{\Gamma_T} v$.
 (c) Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be smooth, convex. Assume that u is smooth and solves the heat equation in U_T . Prove that $v := F(u)$ then is a subsolution.
 (d) Prove that $v = |Du|^2 + u_t^2$ is a subsolution whenever u is smooth and solves the heat equation in U_T .
2. Problem 2.5.21 in Evans. As always when smoothness is not explicitly stated, assume all functions considered are smooth.

Hint: For any vector field $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, recall that $\text{curl} A(x) := \nabla \times A(x)$ and the curl-curl identity tells us that under sufficient regularity,

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \Delta A,$$

(where ∇ here plays the same role as the gradient operator D does normally in Evans' book).

3. Problem 3.5.1 in Evans.

Solution:

1. (a) Let $\bar{v}(y, s) := v(y + x, s + t)$. Then

$$\frac{1}{r^n} \iint_{E(x,t;r)} v(y, s) \frac{|x - y|^2}{(t - s)^2} dy ds = \frac{1}{r^n} \iint_{E(0,0;r)} \bar{v}(y, s) \frac{|y|^2}{s^2} dy ds = \varphi(r)$$

And since $v_s \leq \Delta v$ implies that $\bar{v}_s \leq \Delta \bar{v}$, the equation for φ' (with $\bar{v}$ instead of v) yields

$$\begin{aligned} \varphi'(r) &\geq \sum_{i=1}^n \frac{1}{r^{n+1}} \iint_{E(r)} -4n\Delta \bar{v} \psi - \frac{2n}{s} \bar{v}_{y_i} y_i dy ds \\ &= \sum_{i=1}^n \frac{1}{r^{n+1}} \iint_{E(r)} 4n\bar{v}_{y_i} \psi_{y_i} - \frac{2n}{s} \bar{v}_{y_i} y_i dy ds \\ &= 0, \end{aligned}$$

where the last equality follows by the definition of ψ . We conclude by the smoothness of $\bar{v}$ that

$$\begin{aligned} \frac{1}{4r^n} \iint_{E(x,t;r)} v(y, s) \frac{|x - y|^2}{(t - s)^2} dy ds &\geq \lim_{\rho \rightarrow 0} \varphi(\rho) \\ &= \bar{v}(0, 0) \lim_{\rho \rightarrow 0} \frac{1}{4\rho^n} \iint_{E(\rho)} \frac{|y|^2}{s^2} dy ds \\ &= \bar{v}(0, 0) = v(x, t). \end{aligned}$$

- (b) Let $M = \max_{\bar{U}_T} v$ and suppose there exists an “interior” point $(x, t) \in U_T$ such that $v(x, t) = M$. Since there exists an $E(x, t; r) \subset U_T$ with $r > 0$ (really for any point in U_T), we conclude that

$$M = v(x, t) \leq M \frac{1}{4r^n} \iint_{E(x,t;r)} \frac{|x - y|^2}{(t - s)^2} = M.$$

Hence, v is constant on $E(x, t; r)$, we conclude that the set $A = \{(y, s) \in U_t \mid v = M\}$ is both open and relatively closed in U_t , which, since U is connected, implies that $A \supset U_t$. Hence if v maximizer $(x, t) \in U_T$, then v is a constant function in $\bar{U}_t$, which implies that

$$\sup_{U_T} v \leq \max_{\Gamma_T} v.$$

- (c) Using that $F' \geq 0$ and $u_t = \Delta u$, we obtain that $\partial_t F(u) = F'(u)u_t = F'(u)\Delta u$. And as $\Delta F(u) = (F'(u))^2 |Du|^2 + F''(u)\Delta u$, we conclude that $\partial_t F(u) \leq \Delta F(u)$.

- (d) By direct calculation

$$v_t = 2Du \cdot Du_t + 2u_t u_{tt} = 2Du \cdot D(\Delta u) + 2u_t (\Delta u)_t$$

and

$$\Delta v = 2Du \cdot D(\Delta u) + 2 \sum_{i,j=1}^n (u_{x_i x_j})^2 + 2Du_t \cdot Du_t + 2u_t(\Delta u)_t \geq 2Du \cdot D(\Delta u) + 2u_t(\Delta u)_t = v_t.$$

2. (a) From the equations we have that

$$B_{tt} = -\nabla \times E_t = -\nabla \times (\nabla \times B) = \Delta B - \underbrace{\nabla(\nabla \cdot B)}_{=0} = \Delta B.$$

and, similarly,

$$E_{tt} = \nabla \times B_t = -\nabla \times (\nabla \times E) = \Delta E.$$

(b) Noting that $\text{div} = \nabla \cdot = D \cdot$, Δ and ∂_{tt} all commute and are linear, we have for the first problem

$$w_{tt} = \mu \text{div}(\Delta u) + (\lambda + \mu) \underbrace{D \cdot D(\text{div}(u))}_{=\Delta} = (\lambda + 2\mu) \Delta \text{div}(u) = (\lambda + 2\mu) \Delta w$$

For the second problem, here written without boldface: $w = \text{curl} u$, using that Δ and curl and ∂_{tt} commute and that all operators are linear,

$$w_{tt} = \nabla \times u_{tt} = \mu \nabla \times (\Delta u) + (\mu + \lambda) \nabla \times \nabla(\nabla \cdot u) = \mu \Delta(\nabla \times u) = \mu \Delta w.$$

The penultimate step follows from the observation

$$\nabla \times \nabla = \begin{bmatrix} \partial_{x_2} \partial_{x_3} - \partial_{x_3} \partial_{x_2} \\ \partial_{x_3} \partial_{x_1} - \partial_{x_1} \partial_{x_3} \\ \partial_{x_1} \partial_{x_2} - \partial_{x_2} \partial_{x_1} \end{bmatrix}$$

so that when applying this operator to any smooth function, e.g. $\nabla \cdot u$,

$$\nabla \times \nabla(\nabla \cdot u) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

3. $u_t = -H(a)$ and $Du = a$ yields that $u(x, t, a, b)$ solves the PDE, as

$$u_t + H(Du) = -H(a) + H(a) = 0.$$

Moreover,

$$D_{(x,t)(a,b)}^2 u = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & 1 & 0 \\ -H_{a_1} & \cdots & \cdots & -H_{a_n} & 0 \end{bmatrix}$$

and

$$D_{(a,b)}u = \begin{bmatrix} x_1 - tH_{a_1}(a) \\ \vdots \\ x_n - tH_{a_n}(a) \\ 1 \end{bmatrix}$$

shows that the rank of the matrix-valued function $[D_{(x,t)(a,b)}^2 u, D_{(a,b)}u] \in \mathbb{R}^{(n+1) \times (n+2)}$ is $n + 1$ for all (x, t) and all (a, b) in the respective sets (since the determinant of the square matrix consisting of all rows and all but the penultimate column (i.e., the last column of $D_{(x,t)(a,b)}^2 u$) equals 1).

From the definition of complete integrals, note that the rank must be $n + 1$ rather than n in this exercise since we consider points (x, t) in an $n + 1$ dimensional domain.