

MAT4301 – Suggested exercises from Evans – week 43

October 24, 2025

1. Problem 3.5.4
2. Problem 3.5.5 (a) and (c). Note that the solutions you obtain with the method of characteristics may not be global solutions, try to solve the PDE in part (c) on the domain $U = \{x \in \mathbb{R}^2 \mid x_2 \neq 2\}$.
3. Problem 3.5.6 for the setting when $n = 2$. In part (b), the concise hint given in Evans is, show that $\partial_s(v(X(s, x, t), s)J(s, x, t)) = 0$ provided $v(x, t)$ is a solution of the PDE stated.

Hint: writing $X(s, x, t) = (X^1, X^2)$, use the the ODE $X_s(s, x, t) = b(X(s, x, t)) = (b^1(X), b^2(X))$ and the chain rule to show that

$$X_{sx_i}^1 = b_{x_1}^1(X)X_{x_i}^1 + b_{x_2}^1(X)X_{x_i}^2.$$

and similarly for $X_{sx_i}^2$.

Solution:

1. (a) Writing $[p, q] = [Du, u_t]$ with $q = u_t$, and $z = u$, the PDE $u_t + b \cdot Du - f(x, t) = 0$ can be written $F((p, q), z, (x, t)) = q + b \cdot p - f(x, t) = 0$.
Characteristics:

$$\dot{X} = D_p F = b$$

$$\dot{Z} = \left(D_{(x,t)} u(X(t), t) \cdot (\dot{X}, 1) \right) = (P, Q) \cdot D_{(p,q)} F = b \cdot P + Q = f(X(t), t)$$

observing that this is a closed system of ODE containing all the information we need to solve the PDE, we omit the ODE for (P, Q) .

- (b) We see that $X(t) = x^0 + bt$ and

$$Z(t) = z^0 + \int_0^t f(x^0 + bs, s) ds = g(x^0) + \int_0^t f(x^0 + bs, s) ds.$$

Inversion: Given $(x, t) \in \mathbb{R}^n \times (0, \infty)$, we see that $x^0 = x - bt$ yields $x(t) = x$. Hence,

$$u(x, t) = Z(t) = g(x - bt) + \int_0^t f(x + b(s - t), s) ds$$

2. (a) The linear PDE $F(Du, u, x) = x \cdot Du - 2u = 0$ can be written $F(p, z, x) = x \cdot p - 2z = 0$ with closed system of (X, Z) -characteristics

$$\dot{X} = D_p F = X \implies X(t) = x^0 \exp(t)$$

$$\dot{Z} = P \cdot D_p F = 2Z \implies Z(t) = z^0 \exp(2t)$$

Note that since $x_2^0 = 1$ on the boundary $\Gamma = \{x \in \mathbb{R}^2 \mid x_2 = 1\}$, we have that $X_2(t) \in (0, \infty)$ for all times (which implies that MoC can at best solve the PDE on $\mathbb{R} \times (0, \infty)$).

Given $x \in \mathbb{R} \times (0, \infty)$, we seek $x^0 = (x_1^0, 1) \in \Gamma$ and $t \in \mathbb{R}$ such that $X(t) = x$. Solution: $t = \log(x_2)$ and $x_1^0 = e^{-t} x_1$. This yields

$$u(x) = Z(t) = g(x_1 e^{-t}) e^{2t} = g(x_1/x_2) x_2^2$$

(Let us note that on the boundary, the noncharacteristic boundary condition holds as $\nu(x) \cdot D_p F = \nu(x) \cdot x = x_2 \neq 0$ (as $\nu(x) = \pm e_2$ are "outward" pointing normal vectors on Γ (which is not a boundary but a line in the interior of the domain, as we solve the problem)).

- (c) The PDE can be written $F(p, z) = p \cdot (z, 1) - 1 = 0$ with characteristics

$$\dot{X} = D_p F = (Z, 1)$$

$$\dot{Z} = P \cdot D_p F = 1.$$

We once again omit the ODE for P , since the above system is closed. Noting that $x^0 = (x_1^0, x_1^0)$ on the line $\Gamma = \{x \mid x_1 = x_2\}$ where the BC is given, we obtain the solution $Z(t) = z^0 + t = x_1^0/2 + t$,

$$X_1(t) = x_1^0 + \int_0^t Z(s) ds = x_1^0(1 + t/2) + t^2/2$$

and $X_2(t) = x_1^0 + t$.

As $\nu(x^0) = (-1, 1)$ on Γ and $D_p F(p(x^0), g(x^0)) = (x_1/2^0, 1)$, the noncharacteristic boundary condition holds for $x_1^0 \neq 2$.

Inversion: Given $x \in \mathbb{R}^2$, we seek $x_1^0 \in \mathbb{R} \setminus \{2\}$ and $t \in \mathbb{R}$ s.t. $X(t) = x$
Nonlinear system of equations:

$$x_1^0(1 + t/2) + t^2/2 = x_1$$

$$x_1^0 + t = x_2$$

Inserting $x_1^0 = x_2 - t$ into the first equation yields $t(x_2 - 2) = 2(x_1 - x_2)$. This equation has solution $t = 2(x_1 - x_2)/(x_2 - 2)$ when $x_2 \neq 2$ and

$$x_1^0 = x_2 + \frac{2x_2 - 2x_1}{x_2 - 2}.$$

When $x_2 = 2$, then the equation only can be solved if also $x_1 = 2$ for any $t \in \mathbb{R}$, and otherwise it has no solution. We thus restrict

ourselves to solving the problem in the domain that can be reached by characteristics, namely in $U = \{x \in \mathbb{R}^2 \mid x_2 \neq 2\}$, where

$$u(x) = Z(t) = \frac{x_2}{2} + \frac{x_1 - x_2}{x_2 - 2} = \frac{x_1 - 2x_2 + x_2^2/2}{x_2 - 2}.$$

3. (a) Using the hint

$$\begin{aligned} J_s &= \partial_s (X_{x_1}^1 X_{x_2}^2 - X_{x_1}^2 X_{x_2}^1) \\ &= (b_{x_1}^1(X) X_{x_1}^1 + b_{x_2}^1(X) X_{x_1}^2) X_{x_2}^2 + (b_{x_1}^2(X) X_{x_2}^1 + b_{x_2}^2(X) X_{x_2}^2) X_{x_1}^1 \\ &\quad - (b_{x_1}^2(X) X_{x_1}^1 + b_{x_2}^2(X) X_{x_1}^2) X_{x_2}^1 - (b_{x_1}^1(X) X_{x_2}^1 + b_{x_2}^1(X) X_{x_2}^2) X_{x_1}^2 \\ &= (b_{x_1}^1(X) + b_{x_2}^2(X)) (X_{x_1}^1 X_{x_2}^2 - X_{x_1}^2 X_{x_2}^1) \\ &= \operatorname{div}(b(X)) J \end{aligned}$$

(b) Following the hint, suppose that $v(x, t)$ is a solution of $v_t + \operatorname{div}(vb) = 0$ with the given BC $v(x, 0) = g(x)$. Note also that $\operatorname{div}(vb) = Dv \cdot b + v \operatorname{div}(b)$. Then

$$\partial_s (v(X(s, x, t), s) J(s, x, t)) = (Dv \cdot X_s + v_s) J + v J_s = \underbrace{(v_s + Dv \cdot b(X) + v \operatorname{div}(b))}_{=0} J = 0$$

Hence,

$$v(X(t, x, t), s) J(t, x, t) - v(X(0, x, t), s) J(0, x, t) = \int_0^t \partial_s (v(X(s, x, t), s) J(s, x, t)) ds = 0$$

and the result follows from $X(t, x, t) = x$ and $J(t, x, t) = 1$.