

MAT4301 – Suggested exercises from Evans – week 45

November 1, 2025

1. Problem 3.5.9. Assume that the action L is smooth.

Hint: For parts a) and b) one can take inspiration from the proof of Theorem 3.3.1, but note that the “variations/perturbation paths” $\mathbf{y}(s)$ in $\mathbf{w}(s) = \mathbf{x}(s) + \tau \mathbf{y}(s)$ only are constrained at final the time $s = t$ (by $\mathbf{y}(t) = 0$) and has no constraint at the initial time $s = 0$.

2. Problem 3.5.10

3. Problem 3.5.16

Solution:

1. (a) For $\tau \in \mathbb{R}$, let $\mathbf{w}(s, \tau) = \mathbf{x}(s) + \tau \mathbf{y}(s)$ where $\mathbf{y} \in C^2([0, t]; \mathbb{R}^n)$ with $\mathbf{y}(t) = 0$. Then $\mathbf{w}(\cdot, \tau) \in \mathcal{A}$ and the function

$$i(\tau) := I[\mathbf{w}(\cdot, \tau)] = \int_0^t L(\dot{\mathbf{w}}(s), \mathbf{w}(s)) ds$$

(where we have used the shorthand $\mathbf{w}(s) = \mathbf{w}(s, \tau)$ inside the integral), has a minimum at $\tau = 0$. By the smoothness of $\mathbf{w}$ and L , it follows that

$$i'(\tau) = \int_0^t D_v L(\dot{\mathbf{x}} + \tau \dot{\mathbf{y}}, \mathbf{x} + \tau \mathbf{y}) \cdot \dot{\mathbf{y}} + D_x L(\dot{\mathbf{x}} + \tau \dot{\mathbf{y}}, \mathbf{x} + \tau \mathbf{y}) \cdot \mathbf{y} ds,$$

and that

$$\begin{aligned} 0 = i'(0) &= \int_0^t D_v L(\dot{\mathbf{x}}, \mathbf{x}) \cdot \dot{\mathbf{y}} + D_x L(\dot{\mathbf{x}}, \mathbf{x}) \cdot \mathbf{y} ds \\ &= \int_0^t \left[-\frac{d}{ds} D_v L(\dot{\mathbf{x}}, \mathbf{x}) + D_x L(\dot{\mathbf{x}}, \mathbf{x}) \right] \cdot \mathbf{y} ds - D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) \cdot \mathbf{y}(0) \end{aligned} \tag{1}$$

holds for all C^2 variations $\mathbf{y}$ with $\mathbf{y}(t) = 0$. That it in particular must hold for all $\mathbf{y} \in C^2([0, t])$ with $\mathbf{y}(0) = \mathbf{y}(t) = 0$, implies

$$-\frac{d}{ds} D_v L(\dot{\mathbf{x}}, \mathbf{x}) + D_x L(\dot{\mathbf{x}}, \mathbf{x}) = 0 \quad \text{in } (0, t)$$

(as then the last term in the last line of (1) vanishes). Let us also recall the one BC that was given from the start: $\mathbf{x}(t) = x$.

- (b) By the recently derived Euler-Lagrange ODE, what remains of (1) simplifies to

$$D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) \cdot \mathbf{z} = 0 \quad \forall \mathbf{z} \in \mathbb{R}^n,$$

when writing $\mathbf{z} := \mathbf{y}(0)$. This implies a second BC: $D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) = 0$.

- (c) We proceed as in part (a), with $\mathbf{w}(s, \tau) = \mathbf{x}(s) + \tau \mathbf{y}(s)$ for $\tau \in \mathbb{R}$, but now with the function

$$i(\tau) = I(\mathbf{w}) = \int_0^t L(\dot{\mathbf{w}}(s), \mathbf{w}(s)) ds + g(\mathbf{w}(0))$$

By the smoothness of $\mathbf{w}$, L , and (let us also assume) g , it follows that

$$i'(\tau) = \int_0^t D_v L(\dot{\mathbf{x}} + \tau \dot{\mathbf{y}}, \mathbf{x} + \tau \mathbf{y}) \cdot \dot{\mathbf{y}} + D_x L(\dot{\mathbf{x}} + \tau \dot{\mathbf{y}}, \mathbf{x} + \tau \mathbf{y}) \cdot \mathbf{y} ds + Dg(\mathbf{x}(0) + \tau \mathbf{y}(0)) \cdot \mathbf{y}(0).$$

As before, the function has by assumption a minimum at $\tau = 0$, so that

$$\begin{aligned} 0 &= i'(0) \\ &= \int_0^t \left[-\frac{d}{ds} D_v L(\dot{\mathbf{x}}, \mathbf{x}) + D_x L(\dot{\mathbf{x}}, \mathbf{x}) \right] \cdot \mathbf{y} ds + \left[Dg(\mathbf{x}(0)) - D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) \right] \cdot \mathbf{y}(0). \end{aligned}$$

We conclude (as in a) and b)) that $\mathbf{x}$ solves the Euler-Lagrange ODE

$$-\frac{d}{ds} D_v L(\dot{\mathbf{x}}, \mathbf{x}) + D_x L(\dot{\mathbf{x}}, \mathbf{x}) = 0 \quad \text{in } (0, t),$$

with the one BC $\mathbf{x}(t) = x$ given, and other BC following from above: $D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) = Dg(\mathbf{x}(0))$.

2. (a) For a given v , p^* is a minimizer of $L(v)$ iff $D_p H(p^*) = v$. In other words, $|p^*|^{r-2} p^* = v$ and

$$L(v) = v \cdot p^* + H(p^*) = (1 - r^{-1})|p^*|^r$$

Using that $s^{-1} = 1 - r^{-1}$ and $|v| = |p^*|^{r-1}$, we arrive at

$$L(v) = v \cdot p^* + H(p^*) = s^{-1}|v|^{r/(r-1)} = s^{-1}|v|^s.$$

- (b) For a given v , $\bar{p}$ is a minimizer of $L(v)$ iff $D_p H(\bar{p}) = v$, and $D_p H(\bar{p}) = A\bar{p} + b$. This tells us that $A\bar{p} + b = v$, so that $\bar{p} = A^{-1}(v - b)$ (as A is invertible), and the solution (by the symmetry of A^{-1})

$$L(v) = v \cdot \bar{p} - H(\bar{p}) = \frac{1}{2} \bar{p}^T A \bar{p} = \frac{1}{2} (v - b)^T A^{-1} (v - b).$$

3. For any given $(x, t) \in \mathbb{R}^n \times (0, \infty)$, let $y^i(x, t)$ be minimizers of the respective Hopf-Lax formulas, meaning

$$\min_y \left[tL\left(\frac{x-y}{t}\right) + g^i(y) \right] = tL\left(\frac{x-y^i(x, t)}{t}\right) + g^i(y^i(x, t)) \quad i = 1, 2.$$

Then

$$\begin{aligned} u^1(x, t) - u^2(x, t) &= \min_y \left[tL\left(\frac{x-y}{t}\right) + g^1(y) \right] - \left(tL\left(\frac{x-y^2(x, t)}{t}\right) + g^2(y^2(x, t)) \right) \\ &\leq tL\left(\frac{x-y^2(x, t)}{t}\right) + g^1(y^2(x, t)) - \left(tL\left(\frac{x-y^2(x, t)}{t}\right) + g^2(y^2(x, t)) \right) \\ &\leq \|g^1 - g^2\|_\infty \end{aligned}$$

Similarly,

$$\begin{aligned} u^1(x, t) - u^2(x, t) &\geq tL\left(\frac{x-y^1(x, t)}{t}\right) + g^1(y^1(x, t)) - \left(tL\left(\frac{x-y^1(x, t)}{t}\right) + g^2(y^1(x, t)) \right) \\ &\geq -\|g^1 - g^2\|_\infty. \end{aligned}$$

This implies that

$$|u^1(x, t) - u^2(x, t)| \leq \|g^1 - g^2\|_\infty.$$