

MAT4301 – Suggested exercises from Evans – week 46

November 13, 2025

1. Consider the PDE

$$\begin{aligned} u_t + F(u)_x &= 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(\cdot, 0) &= g & \text{in } \mathbb{R}, \end{aligned} \tag{1}$$

where $F \in C^2(\mathbb{R})$ is uniformly convex ($F'' \geq \theta > 0$) and $g \in L^\infty(\mathbb{R})$. We recall from the lectures that the Lax-Oleinik formula

$$u(x, t) = \begin{cases} g(x) & \text{if } t = 0 \\ G\left(\frac{x - y(x, t)}{t}\right) & \text{if } t > 0 \end{cases}$$

with $G = (F')^{-1}$ and $y(x, t)$ denoting a minimizer of

$$z \mapsto tL\left(\frac{x - z}{t}\right) + \underbrace{\int_0^z g(s) ds}_{=: h(z)},$$

is an integral solution of (1) (the proof of Theorem 3.4.2 details why it is an integral solution).

Assume there exists a $C > 0$ s.t. $|G(x) - G(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}$. Show that the candidate solution then satisfies

$$u(x + z, t) \leq u(x, t) + C \frac{z}{t} \quad \forall (x \in \mathbb{R}, \quad t, z > 0),$$

and thus is an entropy solution.

Hint: You may use that $G' \geq 1/\theta > 0$ (as we proved in the lectures), consider $u(x + z, t)$ for $z > 0$ and recall that $y(x + z, t) \geq y(x, t)$

2. Consider the PDE (1) with $F(u) = u^2/2$ and $g(x) = 1_{(-\infty, 0)}(x)$. Show that

$$y(x, t) = \begin{cases} x - t/2 & \text{if } x < t/2 \\ x & \text{if } x \geq t/2 \end{cases}$$

and use find the unique entropy solution. Why is this the unique entropy solution?

3. Consider the PDE (1) with $F(u) = u^2/2$ and $g(x) = 1_{(0, 1)}(x)$.

- (a) By laborious calculations, it can be shown for the minimizer $y(x, t)$ associated to this problem (i.e., with $L(q) = q^2/2$ and $h(z) = z1_{(0,1)}(z)$) that for times $t \in [0, 2]$

$$y(x, t) = \begin{cases} x, & x < 0 \\ 0, & x \in (0, t) \\ x - t, & x \in (t, 1 + t/2) \\ x, & x \geq 1 + t/2 \end{cases}$$

Use this to describe the entropy solution in the time window $[0, 2]$

- (b) By further calculations, one can also show that for $t \geq 2$,

$$y(x, t) = \begin{cases} x, & x < 0 \\ 0, & x \in (0, \sqrt{2t}) \\ x, & x \geq \sqrt{2t}. \end{cases}$$

Use this to determine $u(x, t)$ for $t \geq 2$.

- (c) What "funny" thing happens to the shock wave at $t = 2$?
- (d) Show e.g. by direct calculation that the function $I(t) = \int_{\mathbb{R}} u(x, t) dx$ does not depend on time.
4. Problem 3.4.8 in Evans (show that the implicit solution formula formally is a solution of the conservation law).

Solution:

1.

$$\begin{aligned} u(x + z, t) &= G\left(\frac{x + z - y(x + z, t)}{t}\right) \\ &\leq G\left(\frac{x - y(x + z, t)}{t}\right) + \left| G\left(\frac{x + z - y(x + z, t)}{t}\right) - G\left(\frac{x - y(x + z, t)}{t}\right) \right| \\ &\leq G\left(\frac{x - y(x + z, t)}{t}\right) + C \frac{z}{t} \\ &\leq G\left(\frac{x - y(x, t)}{t}\right) + C \frac{z}{t} \\ &= u(x, t) + C \frac{z}{t} \end{aligned}$$

In the last inequality, we used that since $x \mapsto y(x, t)$ is non-decreasing and $G' > 0$, it holds that $G((x - y(x + z, t))/t) \leq G((x - y(x, t))/t)$.

2. For this problem $F'(u) = u$, so $G(u) = u$. Furthermore,

$$L(q) = F^*(q) = \max_{p \in \mathbb{R}} [pq - F(p)] = q^2/2$$

and $h(z) = z1_{(-\infty, 0)}(z)$. $y(x, t)$ is the minimizer of

$$\varphi(y; x, t) = tL\left(\frac{x - y}{t}\right) + h(y) = \frac{(x - y)^2}{2t} + y1_{(-\infty, 0)}(y).$$

Candidate minimizers: For $y < 0$, $\varphi_y = \frac{y-x}{t} + 1_{(-\infty,0)}(y)$ and $\varphi_y = 0$ yields the candidate

$$y_1(x, t) = x - t 1_{(-\infty,0)}(y_1) = x - t$$

(only valid when $x < t$) with $\varphi(y_1, x, t) = x - t/2$

For $y \geq 0$, we similarly arrive with $\varphi_y = 0$ at the second candidate $y_2(x, t) = x$ (only valid for $x \geq 0$) with $\varphi(y_2; x, t) = 0$ for $x \geq 0$. Comparing the magnitudes of $\varphi(y_1)$ and $\varphi(y_2)$ we arrive at

$$y(x, t) = \begin{cases} x - t, & x < t/2 \\ x, & x > t/2 \end{cases}$$

and

$$u(x, t) = \frac{x - y(x, t)}{t} = \begin{cases} 1, & x < t/2 \\ 0, & x > t/2. \end{cases}$$

Since $g \in L^\infty(\mathbb{R})$ and F is uniformly convex for this problem and G is Lipschitz, Exercise 1 tells us that this is the unique entropy solution (and one can directly verify that the Rankine-Hugoniot condition is satisfied). (G need not be Lipschitz for this to hold, we stated the more general result in the lecture).

3. (a) The Lax-Oleinik formula yields

$$u(x, t) = \frac{x - y(x, t)}{t} = \begin{cases} 0, & x < 0 \\ x/t, & x \in (0, t) \\ 1, & x \in (t, 1 + t/2) \\ 0, & x > 1 + t/2 \end{cases}$$

for times $t \in [0, 2]$.

(b) For $t \geq 2$, the Lax-Oleinik formula yields

$$u(x, t) = \frac{x - y(x, t)}{t} = \begin{cases} 0, & x < 0 \\ x/t, & x \in (0, \sqrt{2t}) \\ 0, & x > \sqrt{2t} \end{cases}$$

(c) We see that for $t \geq 2$, the jump discontinuity along the shock wave/jump discontinuity along the curve $(X(t), t) = (\sqrt{2t}, t)$ has a jump that shrinks in size as time increases: $u_l(X(t), t) - u_r(X(t), t) = X(t)/t - 0 = \sqrt{2}/t$.

Although not part of the exercise, let us also verify that the shock satisfies the Rankine-Hugoniot condition: On the one hand, direct calculation gives that

$$\dot{X} = \frac{d}{dt} \sqrt{2t} = \frac{1}{\sqrt{2t}}$$

While the R-H condition states that

$$\dot{X} = \frac{F(u_l(X(t), t)) - F(u_r(X(t), t))}{u_l(X(t), t) - u_r(X(t), t)} = \frac{1/t - 0}{\sqrt{2}/t} = \frac{1}{\sqrt{2}},$$

OK!

(d) For $t \in [0, 2]$,

$$I(t) = \int_0^t x/t dx + \int_t^{1+t/2} 1 dx = t/2 + (1 - t/2) = 1.$$

For $t \geq 2$,

$$I(t) = \int_0^{\sqrt{2}t} x/t dx = 1.$$

4. For $u(x, t) = g(x - tF'(u(x, t)))$, formal calculations yield $u_t = -g'(x - tF'(u))F'(u)$ and $F(u)_x = F'(u)u_x = F'(u)g'(x - tF'(u))$. The terms cancel so that we get

$$u_t + F(u)_x = 0.$$